VOLUME 1, No. 2 (1956)

NUCLEAR PHYSICS

MONTHLY JOURNAL DEVOTED TO THE EXPERIMENTAL AND THEORETICAL STUDY OF ATOMIC NUCLEI, NUCLEAR FIELDS AND THE FUNDAMENTAL ASPECTS OF COSMIC RADIATION

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NUCLEAR PHYSICS is published in monthly issues. The subscription price of a volume of 700 pages is fl 53.00; 106s; $ 14.00. Postage 8s; $ 1.00.

SUBSCRIPTIONS should be sent to the publishers, NORTH-HOLLAND PUBLISHING COMPANY, P.O.Box 103, Amsterdam or to any subscription-agent. Subscriptions for U.S.A. are to be sent exclusively to INTERSCIENCE PUBLISHERS, INC., 250 Fifth Avenue, New York 1, N.Y.

Printed in the Netherlands.

Nuclear Physics
1 (1956)

THE EXCITATION CURVES OF SOME (n,p) PROCESSES INDUCED BY NEUTRONS OF 12.5-17.5 MeV ENERGY

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Received 6 September 1955

Abstract: The excitation functions of the reactions Mg^{24} (n,p), Si^{28} (n,p), Cl^{37} (n,p), Sr^{88} (n,p), Cu^{63} (n,2n) and Zn^{64} (n,2n) have been measured, using neutrons of energies between 12.5 MeV and 17.5 MeV from the T(d,n) reaction. The significance of the results is discussed from the viewpoint both of the theory of the compound nucleus and a theory of direct interaction.

1. Introduction

The conventional theory of nuclear reactions proceeding through the formation of a compound nucleus in the statistical region has been summarised by Feld *et al.*[1]) and Blatt and Weisskopf [2]). The cross section for such processes may be estimated if certain assumptions are made about the level densities of the residual nucleus.

Paul and Clarke [3]) bombarded a series of elements with 14.5 MeV neutrons, measured the cross-sections of (n,2n), (n,p) and (n,α) processes, and compared the results with the predictions of this statistical model. They found good agreement for the (n,2n) cross-sections, but the (n,p) and (n,α) cross-sections, although agreeing with the theoretical predictions for $A < 40$, exceeded the theoretical predictions by a factor of 10^2 or 10^3 for heavier nuclei. This discrepancy may arise either because of some error in the assumed level densities or because the reaction proceeds by 'direct interaction' without formation of a compound nucleus.

Some information on nuclear level densities has been given by the recent work of Gugelot, [4,5]) who has remarked that his results resolve the discrepancy claimed by Paul and Clarke.

Some theoretical papers (McManus and Sharp [6]), Austern *et al.*[7])) discuss a possible direct interaction mechanism, which would give cross-sections of several millibarns as are observed, and might help to explain any discrepancy for heavier nuclei. It is however difficult to see how this mechanism can account for high (n,α) cross-sections. Austern *et al.* show that the angular distribution from their direct interaction process should be sharply peaked forward, in contrast

to the isotropic pattern predicted by statistical theory. It seems from the results of Schranck *et al.* [8]), who measured the angular distribution of one long-range group in the reaction Fe^{56} (p,p'), that a large part of this cross-section is indeed due to direct interaction.

A study of the excitation function of (n,p) processes might give some insight into the reaction mechanism. With 1.5 MeV deuterons it is possible to obtain neutrons of energy between 12.5 and 17.5 MeV by the reaction T(d,n)He^4, and to determine excitation functions in this energy range by an activation technique. The choice of reactions is limited to those which have a negative Q of more than −3.5 MeV, to prevent complications from the large background of neutrons from the D(d,n) reaction, and whose products have suitable unambiguously assignable half-lives.

We have studied the excitation functions of the reactions Mg^{24} (n,p), Si^{28} (n,p), Cl^{37} (n,p) and Sr^{88} (n,p), together with the reactions Cu^{63} (n,2n) and Zn^{64} (n,2n), all of which satisfy these criteria. In this paper, we shall discuss the results obtained, and their interpretation on the basis both of the statistical theory, and the theory of direct interaction.

2. Experimental Technique and Results

The tritium target T used was obtained from A.E.R.E., Harwell. The tritium had been absorbed onto a thin zirconium layer deposi-

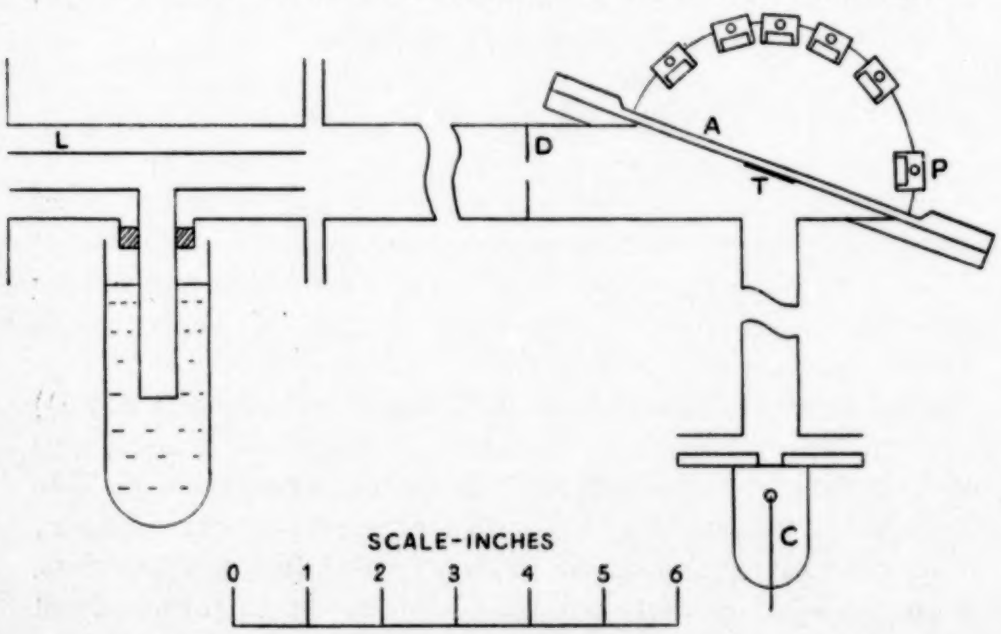

Figure 1. Apparatus

ted on a 1/32" copper disc. This was soldered onto the copper disc A, 6" diameter by 1/8" thick (see fig. 1). It was considered necessary to cool the target to avoid losses of tritium by evaporation; the water cooling system used was designed using the data of Fishenden and Saunders [9]). The deuteron beam from a 1.6 MV electrostatic generator was passed through a liquid air trap L to prevent oil vapour reaching the target, and then through a collimator D onto the target, inclined at 20° to the beam.

The T(d,n)He^4 reaction was monitored by observing the alpha particles at 90° to the beam with a proportional counter C 2' from the target, discriminating against scattered deuterons.

The materials irradiated are listed in table I. Each material was placed in 6 small paxolin cups P of appropriate depth, mounted behind the disc A at suitable angles to the beam, on a circle of radius 5 cm centred at the target spot.

TABLE I

Target Nucleus	Material	Purity	Q MeV	Half-life of Product
Mg^{24}	Element	99.9 %	—4.7	15.0 hours
Si^{28}	Element	99.9 %	—4.0	2.3 minutes
Cl^{37}	NaCl	99.9 %	—3.5	5.0 minutes
Sr^{88}	$SrCO_3$	98 %	—4.2	17.8 minutes
Cu^{63}	Element	99.9 %	—10.9	10.0 minutes
Zn^{64}	Element	99.8 %	—11.8	38 minutes

The target was irradiated with a 20μA beam of 0.68 MeV deuterons for one or two half-lives of the activity required. The alpha yield from the reaction T(d,n) He^4 was monitored at intervals during the irradiation. The specimens were then removed to the counting apparatus. This consisted of a disc on which the six specimens were fitted and which could be rotated under three Geiger counters. Each specimen was placed successively under each counter, and suitably permuted counts were taken for successive intervals until the count was no longer significant. This procedure was repeated for the other materials, and, provided the reaction yield permitted one to work with the lower neutron flux, at deuteron energies of 1.0 and 1.5 MeV.

The decay curves were fitted by weighted least squares to give the initial counting rate. Activities arising from other reactions were negligible, except for strontium, where the 4.5 h activity produced by the $Sr^{88}(n,\alpha)$ reaction was present. This was allowed

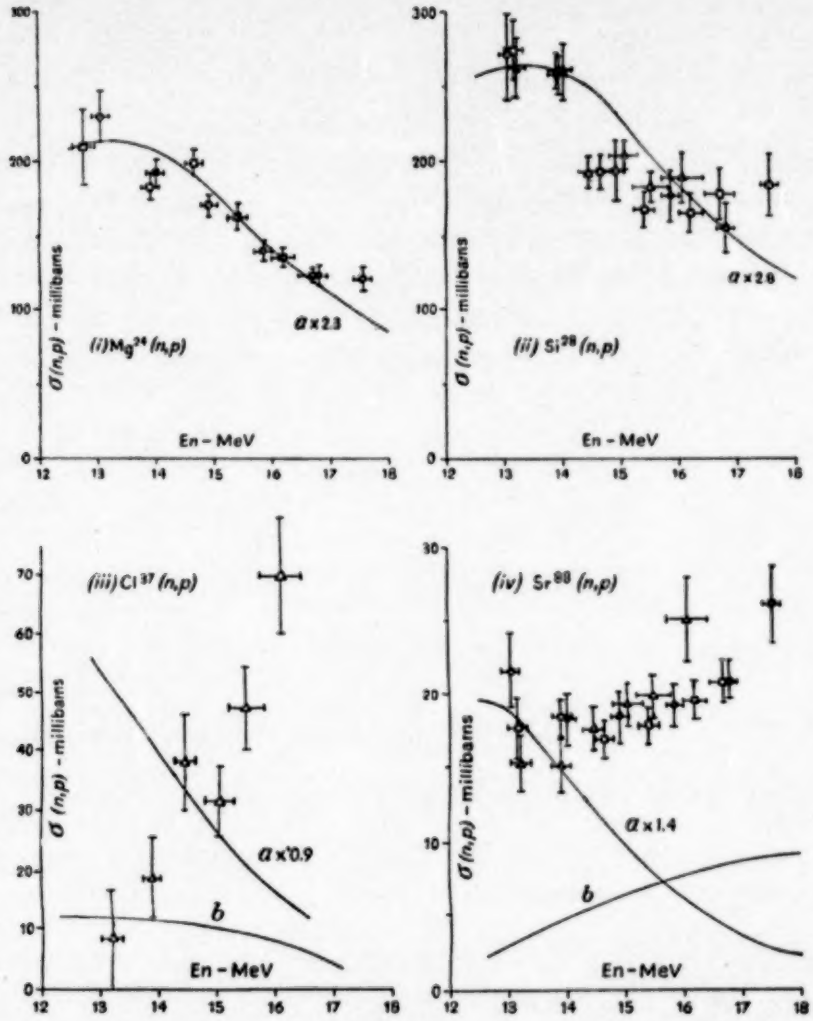

Figure 2. Excitation curves of (n,p) reactions. Absolute cross-sections are taken from the work of Paul and Clarke, and are subject to a standard deviation of 20 %. Triangles represent points observed with $E_d = 0.68$ MeV, squares with $E_d = 1.0$ MeV, and circles with $E_d = 1.5$ MeV.

(i) Mg^{24} (n,p)
(ii) Si^{28} (n,p)
(iii) Cl^{37} (n,p)
(iv) Sr^{88} (n,p)

a. Curve fitted using Gugelot's level densities

b. Direct interaction

for by a suitable technique; since the correction never exceeded 20 % the error introduced was not large.

Corrections were applied for absorption, and for elastic and

inelastic scattering of the neutrons in the copper disc A. A further correction was applied for the target thickness to deuterons, and for carbon deposit on the target. The specimen thickness was in each case chosen to give a β-particle counting rate more than 90 % of that for an infinitely thick specimen. The corrections made for variations of specimen thickness did not exceed 3 %. The error due to all forms of geometrical misalignment did not exceed 5 % and has been included in the quoted results. The technique of counting adopted eliminates any error due to different counter efficiencies.

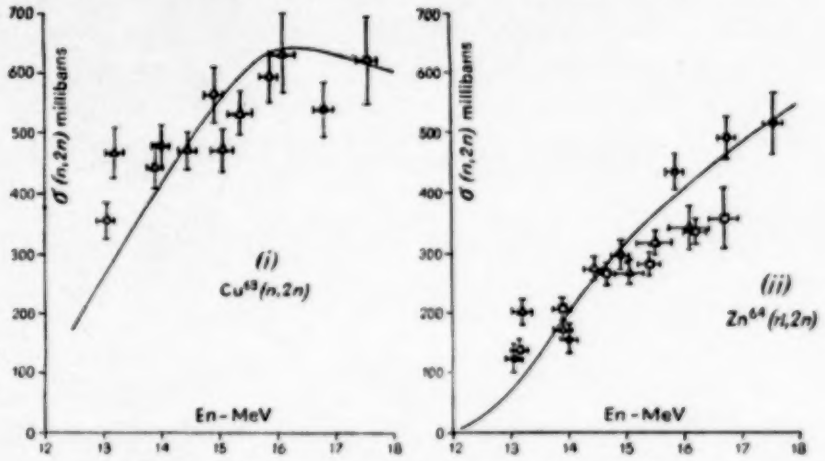

Figure 3. Excitation curves of (n, 2n) reactions
(i) Cu^{63} (n, 2n)
(ii) Zn^{64} (n, 2n)

Excitation curves were obtained from the observed yields using the results of Bame [10]) for the angular distribution of the T(d,n) reaction. Curves obtained for different deuteron energies were normalised together by a process of successive approximations. No significant disagreement between these curves was found so that no serious error has arisen from the correction for scattered neutrons. The absolute scale of cross-sections was taken from the work of Paul and Clarke [3]). Excitation curves for the (n,p) and (n,2n) processes are shown respectively in fig. 2 and 3. The Cu^{63} (n,2n) curve agrees with the results of earlier workers (Brolley *et al.* [11]), Martin and Diven [12]), Fowler and Slye [13])) and provides a further check on the results.

There has been some disagreement among previously reported values of the half-life of Al^{28}. In the Al^{27} (n,γ) reaction, Bartholomew *et al.* [14]) find a half-life of 2.27 $\pm$ 0.02 minutes, and Eklund and Hole [15]) agree with a half-life of 2.30 $\pm$ 0.03 minutes, while Szalay

and Csongor [16]) found that the half-life is 2.07 ± 0.05 minutes in a study of the reaction Mg^{25} (α,p). From the decay curve obtained in the present investigation of Si^{28} (n,p) a half-life of 2.31 ± 0.06 minutes was obtained, agreeing with the later workers.

The errors quoted throughout this work are standard deviations.

3. Theory

3.1 STATISTICAL THEORY.

The analysis made by Weisskopf and others (see for example Blatt and Weisskopf, Chapter 8) shows that the cross-section for the process $X(a,b)Y$ proceeding through the formation of a compound nucleus is

$$\sigma(a,b) = \sigma_c(a)\{F_b'/\Sigma F_c\}, \tag{1}$$

where

$$\begin{aligned} F_b &= \frac{2M_b}{\hbar^2}\int_0^{\varepsilon_{by}} \varepsilon_b \cdot \sigma_c(\varepsilon_b) \cdot W_y(\varepsilon_{by}-\varepsilon_b) \cdot d\varepsilon_b, \\ F_b' &= \frac{2M_b}{\hbar^2}\int_{\varepsilon_{byn}}^{\varepsilon_{by}} \varepsilon_b \cdot \sigma_c(\varepsilon_b) \cdot W_y(\varepsilon_{by}-\varepsilon_b) \cdot d\varepsilon_b. \end{aligned} \tag{2}$$

Here the nomenclature of Blatt and Weisskopf [2]) has been adopted. ε_{by} is the maximum excitation energy of the nucleus Y associated with channel b, and $(\varepsilon_{by}-\varepsilon_{byn})$ is the lowest excitation energy of Y at which the boiling off of a secondary neutron becomes possible.

To compute the F-values it is necessary to know the level densities $W(E) = W(\varepsilon_{by}-\varepsilon_b)$. Blatt and Weisskopf show that, assuming a thermodynamic assembly of the nuclear particles,

$$W(E) = C \exp\,[2(aE)^{\frac{1}{2}}], \tag{3}$$

C and a being fitted to the experimentally observed level densities at about 1 and 8 MeV excitation.

This is a greatly simplified picture of the nucleus. Lang and Le Couteur [17]) have suggested a more suitable semi-empirical formula. Gugelot [4, 5]) investigated the spectra of protons and neutrons found by bombarding various elements with 18 MeV protons. Effectively using formula (2), he deduced level densities for excitations of up to 12 MeV. We have used his results in computing F-values. The level densities are fairly constant for the first few MeV of excitation, and log $W(E)$ thereafter rises more or less linearly with E; a picture very different from that predicted by (3) but which agrees fairly well, above 5 MeV excitation, with the formula of Lang and Le

Couteur, which was used in the necessary extrapolation above 12 MeV.

For the first 4 MeV of excitation, Gugelot found that the scattered protons were peaked in the forward direction, so that his apparent level densities are in fact angle dependent. Thus the reactions he investigated proceed at least in part by some process incompatible with formulae (1)—(3) and to minimise this effect in computing F-values, we took his apparent level densities at backward angles. However, even then, for silver, the apparent level density at low excitations was dependent upon incident proton energy, contrary to the predictions of the statistical model, which must therefore, at least for the heavier elements, be taken with some reserve.

In computing the cross-section for an (n,p) reaction with a Q of -4.0 MeV, with neutrons between 12 and 18 MeV, one compares the level densities in range say 0—8 MeV for the residual nucleus with those for the range 10—16 MeV for the target nucleus. Thus, for example, if one takes level densities obtained for an even-mass nucleus, and applies this to (say) both an even target nucleus and the odd residual nucleus, one might predict too small a cross-section increasing too slowly with energy.

3.2 DIRECT INTERACTION

There is a whole range of possible interactions, from the conventional compound nucleus process to the extreme 'knock-on' mechanism. Austern *et al.* discuss a process in which the incident neutron interacts with that part of the proton wave-function outside the nucleus. The differential cross-section σ(n,p) for such a process is then related to the cross-section σ_{np} for free (n—p) collision, and the (p,p′) and (n,p) cross-sections are closely related. For the simple case of the initial proton and final neutron each in a wholly single-particle S-state, with otherwise identical cores we have

$$\sigma(\mathrm{n,p}) = \sigma_{np} \cdot \frac{k_\mathrm{p}}{k_\mathrm{n}} \cdot \{j_0(Qr_0)\}^2 \left[\int_{r_0}^{\infty} f_\mathrm{n}(r) f_\mathrm{p}(r) r^2 dr\right]^2, \qquad (4)$$

where $\mathbf{Q} = \mathbf{k}_\mathrm{n} - \mathbf{k}_\mathrm{p}$ the vector difference of free neutron and proton wave numbers, is strongly angle-dependent, r_0 is the nuclear radius and f_n and f_p are the bound neutron and proton wave-functions.

In the present computations f_n and f_p were assumed to be of the form $\exp(-kr)/r$ and a term was included in the integrand to allow

for the penetrability of the emergent proton wave. One finds a total cross-section rising from the threshold to a maximum of the order of a millibarn, and then falling off roughly as $1/E^{3/2}$.

If there are N states in the residual nucleus between the ground state (energy $-B_n$) and the state of zero binding energy, and if the computed cross-section for transition from a proton state p to the level t is σ_{pt}, it is not true to say that the total cross-section is $\Sigma_{pt}\sigma_{pt}$, since the levels t are not pure single particle states, and the form of interaction assumed by Austern *et al.* is essentially a single particle transition.

The following procedure was adopted to give a rough estimate of the probable total cross-section. If the expected single particle level-spacing is Δ, and the level degeneracy P, then there are B_nP/Δ possible single particle levels, each of which could by itself be associated with a cross-section given by (4). These single particles are coupled to the others, giving the N levels observed. For a given type of initial proton state the proportion of each of the N levels with a wave-function such that this type of direct interaction is possible is very roughly $B_nP/\Delta N$. A reasonable estimate of the cross-section is therefore

$$\sigma(\mathrm{n,p}) = \frac{B_nP}{\Delta N}\Sigma_{pt}\sigma_{pt}. \qquad (5)$$

For simplicity it was assumed that (4) applied to transitions between levels whatever their l, and that only the nucleons in the outermost shells were involved — the cross-section diminishes rapidly with increasing binding energy because of penetrability effects. This procedure gives estimates which are probably correct to within a factor of 5.

4. Comparison with Experiment

4.1. ABSOLUTE CROSS-SECTIONS AT 14.5 MeV

The F-values computed as described have been used to calculate σ(n,p) and σ(n,α) for all the reactions observed by Paul and Clarke. In no case do σ (observed) and σ (calculated) differ by a factor of more than 10. That is, a statistical theory based on Gugelot's level densities suffices to explain the observed cross-sections, while the level densities either of Blatt and Weisskopf or of Lang and Le Couteur predict much too low cross-sections for the heavier elements. However, the level densities deduced from (p,p′) scattering in heavy

elements are artificially high at low excitations because of some direct interaction effects. If one assumes that only the type of direct interaction described by Austern *et al.* takes place, then since the free (p,p) and (n,p) differential scattering cross-sections are comparable, and not strongly energy-dependent, and if the energy of the long range protons is not much less than the potential barrier, then the insertion of the apparent level densities in the statistical formulae will automatically include the effects of direct interaction. Thus the apparent agreement is not a critical test, but might perhaps be taken as evidence that the (p,p') and (n,p) direct interaction processes are not very different. The agreement with the (n,α) cross-sections is rather surprising since the observed (α,p) spectra do not show the same predominance of long-range particles (Gugelot [18])).

4.2 (n,2n) EXCITATION CURVES

The form expected of the (n,2n) excitation curve on the basis of statistical theory was calculated from the observed level densities. The curve rises from zero at the threshold to $\approx \pi R^2$ (in the absence of competing processes) in an energy range of the order of the nuclear temperature. Thus a curve based on any one of a wide number of reasonable assumptions will fit the results.

Because of competing (n,p) and (n,α) processes, the curve will not rise to πR^2. Allowance has been made for the competing processes, and it was found necessary to scale both the curves down by a factor of 0.6 to fit the observed results. The F_p and F_α values are not sufficiently well known for this factor to be significant.

4.3 (n,p) EXCITATION CURVES

Mg^{24}(n,p) and Si^{28}(n,p). For each of these the observed cross-section at 12.5 MeV is about 250 mbn, and drops steadily to about 100 mbn at 18 MeV. The direct interaction theory predicts cross-sections of less than 5 mbn. Most of the reaction therefore probably occurs through the formation of the compound nucleus. In fig. 2, curves are fitted using F-values deduced from the observed level densities in Al^{27} (odd mass.) The curves have in each case been adjusted by arbitrary factors indicated on the diagram; the F-values are not known sufficiently well for this to be significant. The Al^{27} level densities account for the observed curves surprisingly well, since the target nucleus is in each case even and the residual nucleus odd. One might therefore have expected the experimental curve

to fall off more slowly with energy than the theoretically predicted curve. However, at higher energies the shape of the curves is modified by the competing (n,α) process, the cross-section for which cannot accurately be predicted.

Cl^{37}(n,p). The low yield made it possible to investigate the excitation curve only in the region 13—16 MeV. There is a significant rise in yield from 10 mbn to 70 mbn in this range. The knock-on process should yield cross-sections of about 10 mbn, which fall off with increasing energy. To estimate F-values for this reaction, a suitable mean was taken of the observed level densities for Al^{27} and Fe^{56}. The statistical excitation function thus predicted also falls off with energy. The 'odd-even effect' should not in this case be very great, and the discrepancy may suggest that the extrapolation of Gugelot's level densities in this case increases too rapidly with energy, or that other forms of direct interaction are important.

Sr^{88}(n,p). The cross-section is steady up to about 15 MeV, and thereafter rises slightly. Taking a level density as a mean between those observed in silver and copper (both odd mass nuclei), the excitation curve predicted on the statistical theory falls off too sharply with energy. As for Mg^{24} and Si^{28}, one might indeed expect the experimental curve to fall less sharply, though the discrepancy might be taken as further evidence that the extrapolation of Gugelot's apparent level densities rises too steeply with energy. The situation is complicated by the knock-on process. The calculated cross-section for this mechanism rises, because of penetrability effects, from 1 mbn at 12 MeV to 10 mbn at 18 MeV. Thus a substantial part of the observed cross-section at 18 MeV is probably due to this process, which may account for the slight rise in yield at higher energies.

5. Conclusions

Excitation functions have been measured for the reactions Mg^{24}(n,p), Si^{28}(n,p), Cl^{37}(n,p), Sr^{88}(n,p), Cu^{63}(n,2n) and Zn^{64}(n,2n) using neutrons of energy between 12.5 and 17.5 MeV. The (n,2n) excitation curves agree reasonably well with the predictions of statistical theory. The Mg^{24}(n,p) and Si^{28}(n,p) excitation curves agree surprisingly well with the predictions of a statistical theory based on the observed level densities, but there is a considerable discrepancy for the other two reactions. In these cases the discrepancies might be removed by supposing that the technique of

extrapolating Gugelot's level densities above 12 MeV predicts a level density increasing too rapidly with energy.

The discrepancies might also be explained by a direct interaction. This agrees with the angular distribution measurements of Schranck *et al.* in $Fe^{56}(p,p')$. It is of course possible that the (p,p') and (n,p) processes have entirely different mechanism, but since the level densities deduced from (p,p') processes explain (n,p) cross sections moderately well, any such difference is probably not serious.

The easily observable excitation curves of (n,p) processes do not give a critical test between the conventional compound nuclear process and an extreme knock-on process. This is because of inadequate knowledge of nuclear level densities; further investigation of these might be helpful, but it is difficult to allow for the effects of direct interaction in such work. Moreover, a systematic measurement of cross-sections at one neutron energy as in the work of Paul and Clarke will not readily give a critical test, which apparently can be achieved only by a measurement of angular distributions.

The authors wish to express their sincere thanks to Professor S. Devons for initially suggesting the problem, and for much helpful discussion during the course of the work.

Thanks are also due to Dr. M. J. Poole, for information on the use of zirconium tritide targets, and to Dr. S. J. Bame, who informed us of his results on the angular distribution of the $T(d,n)He^4$ reaction.

One of us (PHW) is indebted to the Department of Scientific and Industrial Research for a maintenance grant during the course of this work.

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Nuclear Physics
1 (1956)

LES CONDITIONS D'EXISTENCE DES SPECTRES DE ROTATION NUCLÉAIRES

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Reçu le 8 juillet 1955

A la mémoire de J. Podolanski

Abstract: Within the framework of the unified model, a perturbation method is devised which allows the study of nuclear rotational spectra under quite general conditions, in particular without assuming that the nucleons coupled to the core are on well defined orbitals. It is possible to treat cases where the core is asymmetric or possesses an axis of symmetry and to show that nuclear rotational spectra depend on two degrees of freedom if the core is asymmetric and one if the core has an axis of symmetry. The wave functions of the system are studied in detail.

1. Introduction

Les noyaux atomiques moyens et lourds ayant une grande déformation permanente présentent un spectre caractéristique de niveaux de rotation. Cette propriété a été mise en évidence, pour la première fois, par Bohr et Mottelson [1]) et constitue, à l'heure actuelle, l'argument le plus fort en faveur du modèle dit unifié de la structure nucléaire.

La méthode suivie par les précédents auteurs [1, 2]) n'épuise toutefois pas les possibilités du modèle simple constitué par un core C couplé fortement à N nucléons. Le travail qui suit vise à montrer que les niveaux de rotation nucléaires existent dans des conditions plus générales que celles indiquées par A. Bohr et qu'en fait on doit distinguer deux types de spectres de rotation suivant que le core est asymétrique ou possède un axe de symétrie.

Les paragraphes 2 et 3 rappellent comment on obtient l'hamiltonien du modèle unifié dans un système d'axes mobiles convenable. Au § 4 on étudie en détail l'allure des fonctions d'ondes dans le cas général où on ne fait pas l'hypothèse que les N nucléons sont caractérisés par des orbitales bien définies. Le paragraphe 5 rappelle comment on obtient l'équation utilisée par Bohr et Mottelson et

analyse ses conditions de validité. Enfin le paragraphe 6 décrit une méthode appropriée pour l'obtention des spectres de rotation dans le cas de cores asymétriques, tandis qu'au § 7 un raisonnement similaire est repris pour un core ayant un axe de symétrie.

2. Variables dynamiques, hamiltonien et moment angulaire du système

Nous considérons le modèle simple de N nucléons a, b, ... couplés au core C. Leurs coordonnées dans le système K_i du laboratoire sont $x_{ai}, x_{bj} \ldots$, leurs moments conjugués $p_{ai}, p_{bj}, \ldots$ et l'hamiltonien H_p relatif à ces particules est

$$H_p = \sum_{ai} \frac{p_{ai}^2}{2m} + \sum_a V(r_a) + \hbar^2 \sum_{ai} \xi(r_a) l_{ai} s_{ai} + \sum_{a>b} V(a,b) \quad (i = x, y, z).$$

Dans H_p, m est la masse des nucléons, $V(r_a)$ un potentiel central, le terme en $\xi(r_a)$ caractérisant un couplage entre le spin $\mathbf{s}_a$ et le moment orbital $\mathbf{l}_a$ de la $a^{\text{ème}}$ particule; $V(a,b)$ représente une interaction entre les nucléons a, et b.

Pour caractériser le core C, nous suivrons une méthode due à Wentzel [3]), équivalente à celle de Bohr [2]), pour des déformations d'ordre 2. Considérons une goutte de matière nucléaire à symétrie sphérique et déformons-la linéairement de telle sorte que chaque point de coordonnée x_{0i} vienne en x_i défini par

$$x_i = x_{0i} + \sum_j \psi_{ij}(t) x_{0j}. \tag{1}$$

Le tenseur $\psi_{ij}(t)$, fonction du temps t, est:

— symétrique pour les mouvements irrotationnels;

— de trace nulle si la matière nucléaire est incompressible.

Cette dernière condition n'étant pas nécessaire pour la suite du développement, nous poserons

$$\psi = \sum_i \psi_{ii} \tag{2}$$

et nous supposerons $\psi \neq 0$.

Pour construire l'hamiltonien du core on considère les variables conjuguées des ψ_{ij}:

$$\Pi_{ij} = -i\hbar \partial / \partial \psi_{ij}.$$

Il est plus commode d'introduire des variables π_{ij} telles que

$$\pi_{ij} = -i\hbar(1 + \delta_{ij}) \partial / \partial \psi_{ij},$$

dont les règles de commutation avec les ψ_{ij} sont

$$[\psi_{ij}, \pi_{i'j'}] = i\hbar(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'}).$$

L'hamiltonien du core est de la forme [3])

$$H_C = \frac{1}{4B}\sum_{ij}\pi_{ij}^2 + \frac{C}{2}\sum_{ij}\psi_{ij}^2 + \frac{C'}{2}\psi^2, \tag{3}$$

où B, C et C' sont des constantes caractéristiques. Le premier terme de (3) est l'opérateur énergie cinétique du core, le second l'énergie potentielle et le troisième permet de tenir compte de la condition d'incompressibilité si on le désire.

L'hamiltonien d'interaction H_{int} entre le core et les nucléons est choisi du type statique:

$$H_{\text{int}} = -K(r)\sum_{ij}\psi_{ij}\sum_{a}(x_{ai}x_{bj} - \tfrac{1}{3}\delta_{ij}r_a^2). \tag{4}$$

L'hamiltonien du système total est

$$H = H_p + H_C + H_{\text{int}} \tag{5}$$

et permet d'écrire l'équation de Schrödinger du système total

$$(H - E)\langle x_{ai}, \psi_{ij}|\tau EIM\rangle = 0. \tag{6}$$

Dans (6), I est la valeur propre du moment total $\mathbf{I}^2$ du système, M la valeur propre de I_z, tous les moments angulaires étant évalués en unités $\hbar$; τ représente d'autres nombres quantiques nécessaires pour déterminer l'état complètement.

Les composantes I_i de $\mathbf{I}$ sont la somme des moments angulaires Q_i du core et J_i des nucléons:

$$I_i = Q_i + J_i \tag{7}$$

avec

$$\hbar Q_{i_\Delta k} = \sum_j(\psi_{ij}\pi_{kj} - \psi_{kj}\pi_{ij}).$$

On a les règles de commutation

$$[Q_i, Q_j] = iQ_{i_\Delta j},\ [J_i, J_j] = iJ_{i_\Delta j},\ [Q_i, J_j] = 0,\ [I_i, I_j] = iI_{i_\Delta j}. \tag{8}$$

3. Passage aux axes principaux du core

Dans le cas du couplage fort, le core a une déformation ellipsoïdale permanente et il est commode d'exprimer toutes les variables dynamiques dans un système de références K_α lié aux axes principaux de C. Par la suite, les vecteurs, tenseurs . . . avec indices grecs se rapporteront à ce système mobile K_α, tandis que les vecteurs,

tenseurs . . . du système du laboratoire K_i auront des indices latins.

Soient $S_{i\alpha}(\theta_u)$ les cosinus directeurs définissant le passage de K_i à K_α, ces deux référentiels étant repérés l'un par rapport à l'autre par les angles d'Euler θ_u. On a

$$\sum_i S_{i\alpha} S_{i\beta} = \delta_{\alpha\beta}, \quad \sum_\alpha S_{i\alpha} S_{k\alpha} = \delta_{ik} \quad (i = x, y, z \quad \alpha = 1, 2, 3). \tag{9}$$

Dans le système K_α, le tenseur transformé des ψ_{ij} est diagonal par définition:

$$\psi_{\mu\nu} = \sum_{ij} S_{i\mu} S_{j\nu} \psi_{ij} = q_\mu \delta_{\mu\nu}.$$

a. TRANSFORMATION DES MOMENTS ANGULAIRES

Les $S_{i\alpha}$ étant définis à partir des degrés de liberté ψ_{ij} et non des x_{ai}, on a tout de suite

$$[J_i, S_{k\mu}] = 0, \tag{10}$$

tandis que $S_{k\mu}$ forme † un vecteur de composantes $k = x, y, z$ qui ne commute pas avec Q_i :

$$[Q_i, S_{k\mu}] = iS_{i_\wedge k,\mu}. \tag{11}$$

Grace aux relations (10)et (11), on peut calculer les commutateurs entre les diverses composantes des vecteurs **I**, **J**, **Q** dans le système mobile. On trouve

$$\begin{aligned} [Q_\alpha, Q_\beta] &= -iQ_{\alpha_\wedge\beta}, \quad [J_\alpha, J_\beta] = +iJ_{\alpha_\wedge\beta}, \\ [Q_\alpha, J_\beta] &= -iJ_{\alpha_\wedge\beta}, \quad [I_\alpha, I_\beta] = -iI_{\alpha_\wedge\beta}, \end{aligned} \tag{8'}$$

qui sont les relations transformées de (8). On peut y ajouter

$$[I_\alpha, J_\beta] = 0, \tag{8''}$$

indiquant que, dans le système K_α, les composantes de **I** et de **J** commutent. Les relations (8') et (8'') peuvent être déduites de l'addition vectorielle

$$-Q_\alpha = -I_\alpha + J_\alpha, \tag{7'}$$

où $-Q_\alpha$, $-I_\alpha$, $+J_\alpha$ jouent respectivement les rôles de I_i, Q_i, J_i de (7) dans le système fixe. Le changement de signe dans (7') devant Q_α, I_α fait que le facteur i apparaissant pour certains commutateurs est toujours précédé du signe $+$ [4]).

† Cf. E. U. Condon and G. H. Shortley, Theory of atomic spectra, Cambridge Univ. Press (1951), p. 59.

b. TRANSFORMATION DE L'HAMILTONIEN

La transformation de l'hamiltonien H se fait en exprimant les variables dynamiques x_{ai}, S_{ai}, ψ_{ij} en fonction des $x_{a\alpha}$, $S_{a\alpha}$, q_μ, θ_ν. On a également intéret à remplacer les q_μ par:

$$q_\mu = \frac{q_0}{\sqrt{3}} + \sqrt{\tfrac{2}{3}}\beta\cos\left(\gamma + \frac{2\pi}{3}\mu\right), \tag{12}$$

où q_0 caractérise la compressibilité du core ($\psi = \sqrt{3}q_0$), β et γ la déformation de celui-ci.

L'hamiltonien H s'écrit en fonction des nouvelles variables (cf. [2]) [3]) et Appendice)

$$H = H_p + H_0 + T_{\text{vib}} + V + T_{\text{rot}} + H_{\text{int}}, \tag{13}$$

où

$$H_p = \sum_{a\alpha}\frac{p_{a\alpha}^2}{2m} + \sum_a V(r_a) + \hbar^2\sum_{a\alpha}\xi(r_a)l_{a\alpha}s_{a\alpha} + \sum_{a>b}V(a,b), \tag{14}$$

$$\left.\begin{aligned} H_0 &= -\frac{1}{2B}\frac{\partial^2}{\partial q_0^2} + \frac{C+3C'}{2}q_0^2, \\ T_{\text{vib}} &= -\frac{1}{2B}\left\{\frac{\partial^2}{\partial\beta^2} + \frac{4}{\beta}\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\left(\frac{\partial^2}{\partial\gamma^2} + 3\cot 3\gamma\frac{\partial}{\partial\gamma}\right)\right\}, \\ V &= \tfrac{1}{2}C\beta^2, \end{aligned}\right\} \tag{15}$$

$$T_{\text{rot}} = \frac{\hbar^2}{8B\beta^2}\sum_\mu\frac{(I_\mu - J_\mu)^2}{\sin^2\left(\gamma - \frac{2\pi}{3}\mu\right)}, \tag{16}$$

$$H_{\text{int}} = -\frac{1}{\sqrt{6}}\sum_a r_a^2 K(r_a)\beta[(3n_{a3}^2 - 1)\cos\gamma + \sqrt{3}(n_{a2}^2 - n_{a1}^2)\sin\gamma] \tag{17}$$

($\mathbf{n}$ vecteur unité dans la direction $\mathbf{r}_a$).

On voit que H_p ne contient que les variables dynamiques des nucléons, tandis que H_0, T_{vib} et V se rapportent au core. Les termes d'interaction core-nucléons sont le terme T_{rot} (appelé aussi énergie cinétique du core) et H_{int}. Le point important, mis en évidence par Wentzel [3]), est que le mouvement des ondes de compressibilité est séparable† du reste de l'hamiltonien. Nous profiterons de cette circonstance pour laisser de côté H_0 dans ce qui suit, où l'on s'intéresse aux niveaux peu excités des noyaux.

† Cette séparation n'existe que pour les déformations linéaires du type (1) (Communication privée du Prof. Rosenfeld).

4. Les fonctions d'onde dans le système mobile

Nous cherchons à écrire les fonctions d'onde $\langle x_{ai}\psi_{ij}|\tau EJM\rangle$ de (6) dans le système mobile K_α. On sait que pour une rotation déterminée du système de référence K_i, les fonctions d'onde $\langle x_{ai}\psi_{ij}|\tau EJM\rangle$ peuvent être regardées comme des combinaisons linéaires de fonctions d'onde $\langle x_{a\alpha}\psi_{\mu\nu}|\tau EIK\rangle$ où K est la valeur propre de la composante $\alpha = 3$ du vecteur $\mathbf{I}$:

$$\langle x_{ai}\psi_{ij}|\tau EIM\rangle = \sum_{K=-I}^{I} D^I_{MK}(\theta)\langle x_{a\alpha}\psi_{\mu\nu}|\tau EIK\rangle, \tag{18}$$

Les matrices $D^I_{MK}(\theta)$ sont les représentations du rang $2I+1$ du groupe des rotations.

Dans le cas du passage de K_i à K_α, comme le tenseur $\psi_{\mu\nu}$ est astreint à être diagonal, on perd 3 degrés de libertés dans la fonction d'onde $\langle x_{a\alpha}, \psi_{\mu\nu} = q_\mu\delta_{\mu\nu}|\tau EIK\rangle$, mais ceux-ci sont remplacés par les 3 angles d'Euler θ_u. Il en résulte que dans (18) les D^I_{MK} ne sont plus des paramètres, mais des fonctions d'onde vérifiant les conditions d'orthogonalité †

$$\int (d\theta) D^I_{MK}(\theta)^*\, D^{I'}_{M'K'}(\theta) = \frac{8\pi^2}{2I+1}\delta_{II'}\delta_{MM'}\delta_{KK'}. \tag{19}$$

Avec les angles d'Euler usuels $(d\theta) = \sin\theta d\theta d\psi d\varphi$.

Si l'on tient compte du fait que l'on néglige la compressibilité, la fonction d'onde (18) s'écrira

$$\Psi^{\tau EIM} \equiv \langle x_{ai}\psi_{ij}|\tau EIM\rangle = \sqrt{\frac{2I+1}{8\pi^2}} \sum_K D^I_{MK}(\theta) G^{\tau EI}_K(x_{a\alpha}, \beta, \gamma). \tag{20}$$

La fonction d'onde $\psi^{\tau EIM}$ doit être invariante quand on passe du système mobile K_α à un autre système mobile K'_α qui ne diffère de K_α que par la numérotation ou le sens des axes principaux de l'ellipsoïde du core. Cette invariance a été étudiée en détail par Bohr [2]) dans le cas d'un seul nucléon de nombres quantiques bien définis et nous la reprenons ici pour déterminer la forme générale de $\psi^{\tau EIM}$.

a. CORE ASYMÉTRIQUE

La fonction $\Psi^{\tau EIM}$ doit être invariante vis-à-vis du changement de sens de l'axe 3 (et de l'axe 2 pour garder un trièdre droit). Bohr envisage alors une première transformation R_1 renversant le sens

† E. Wigner, Gruppentheorie, Vieweg und Sohn, Braunschweig (1931), p. 110.

des axes mobiles 2 et 3. R_1 ne change ni β ni γ et appliquée aux $D^I_{MK}(\theta)$ donne

$$R_1 D^I_{MK}(\theta) = (-1)^{I+K} D^I_{M-K}(\theta), \quad R_1^2 = 1.$$

De

$$\sum_K D^I_{MK} G^{\tau EI}_K \equiv \sum_K (-1)^{I+K} D^I_{M-K} (R_1 G^{\tau EI}_K)$$

on déduit

$$G^{\tau EI}_{-K} = (-1)^{I+K} R_1 G^{\tau EI}_K. \tag{21}$$

La fonction d'onde $\Psi^{\tau EIM}$ s'écrit alors

$$\Psi^{\tau EIM} = \sum_{K \geqq 0} N_{IK} \{ D^I_{MK} G^{\tau EI}_K + (-1)^{I+K} D^I_{M-K} R_1 G^{\tau EI}_K \}, \tag{22}$$

où N_{IK} est un facteur de normalisation.

La seconde transformation R_2 introduite par Bohr revient à faire tourner les axes mobiles de $\frac{1}{2}\pi$ autour de l'axe 3. Elle ne change pas β, mais transforme γ en $-\gamma$, et

$$R_2 D^I_{MK} = e^{i\frac{\pi}{2}K} D^I_{MK}, \quad R_2^4 = 1.$$

Appliquée à (20), l'opération R_2 conduit à

$$G^{\tau EI}_K = e^{i\frac{\pi}{2}K} R_2 G^{1EI}_K. \tag{23}$$

On voit qu'elle revient à imposer une condition aux fonctions G.

Finalement Bohr utilise une troisième transformation R_3 ($R_3^3 = 1$) qui permute cycliquement les axes 1, 2, 3 et que nous utiliserons ici pour borner la variation de γ au domaine

$$-\frac{\pi}{3} \leqq \gamma \leqq \frac{\pi}{3}.$$

Ces trois transformations R_1, R_2, R_3, combinées convenablement, effectuent toutes les transformations possibles entre les différents trièdres droits K_α issus d'un même référentiel K_i.

Il est intéressant d'analyser plus en détail le comportement des fonctions $G^{\tau EI}_K$ sous l'action des transformations R_1 et R_2. Nous introduisons pour cela les fonctions propres de l'hamiltonien $H_p - \sum_{a>b} V(a,b)$ de (14):

$$\left\{ \sum_{a\alpha} \frac{\hat{p}^2_{a\alpha}}{2m} + \sum_a V(r_a) + \hbar^2 \sum_{a\alpha} \xi(r_a) l_{a\alpha} s_{a\alpha} - E_{\text{ind}} \right\} \langle x_{a\alpha} | \eta J \Omega \rangle = 0.$$

Les fonctions d'ondes $\langle x_{a\alpha} | \eta J \Omega \rangle$ sont de type individuel, η symboli-

sant les divers nombres quantiques usuels de couplage jj: $n_1 l_1 j_1$, $n_2 l_2 j_2$, . . ., $n_N l_N j_N$, l'énergie E_{ind} de l'état et d'autres constantes de mouvement si cela est nécessaire; Ω est la projection sur l'axe mobile 3 du moment total **J** des particules, dans un état où sa longueur est J.

L'action des opérateurs R_1, R_2 sur les fonctions d'ondes précédentes est [2])

$$R_1\langle x_{a\alpha}|\eta J\Omega\rangle = (-1)^{J+\Omega}\langle x_{a\alpha}|\eta J - \Omega\rangle,$$

$$R_2\langle x_{a\alpha}|\eta J\Omega\rangle = e^{-i\frac{\pi}{2}\Omega}\langle x_{a\alpha}|\eta J\Omega\rangle.$$

On peut alors développer les fonctions $G_K^{\tau EI}$ de la façon suivante:

$$G_K^{\tau EI}(x_{a\alpha}, \beta, \gamma) = \sum_{\eta J\Omega} \varphi_{K\Omega}^{\tau EI\eta J}(\beta, \gamma)\langle x_{a\alpha}|\eta J\Omega\rangle. \tag{24}$$

En suivant exactement le même raisonnement que Bohr, on trouve, si l'on applique R_2^2 à la fonction $G_K^{\tau EI}$:

$$\varphi_{K\Omega}^{\tau EI\eta J}(\beta, \gamma) \neq 0, \quad \text{si } K - \Omega = 2\nu \quad (\nu \text{ entier}).$$

Il en résulte que dans (24), *les valeurs possibles de* Ω *varient de* 2 *en* 2. (21) et (23) donnent

$$\begin{aligned} \varphi_{K\Omega}^{\tau EI\eta J}(\beta, \gamma) &= (-1)^{\nu}\varphi_{K\Omega}^{\tau EI\eta J}(\beta, -\gamma) \\ \varphi_{-K-\Omega}^{\tau EI\eta J}(\beta, \gamma) &= (-1)^{I-J}\varphi_{K\Omega}^{\tau EI\eta J}(\beta, \gamma), \end{aligned} \tag{25}$$

ces deux relations étant identiques à celles obtenues dans le cas d'une seule particule de nombres quantiques n, l, j définis, couplée au core.

On remarquera que pour $K = 0$, les conditions (25) restreignent les valeurs possibles de J figurant dans (24). En effet, s'il existe dans (24) un terme en $\Omega = 0$, la seconde relation (25) implique que $I - J$ doit être pair. Le développement de $G_K^{\tau EI}$ pour $K = 0$ devient donc

$$G_0^{\tau EI} = \sum_{\eta J'} \varphi_{00}^{\tau EI\eta J'}\langle x_{a\alpha}|\eta J'0\rangle + \sum_{\substack{\eta J \\ \Omega>0}} \varphi_{0\Omega}^{\tau EI\eta J}\{\langle x_{a\alpha}|\eta J\Omega\rangle + (-1)^{I-J}\langle x_{a\alpha}|\eta J - \Omega\rangle\} \tag{24'}$$

où les valeurs de J' sont telles que $I - J'$ est pair.

b. CORE À UN AXE DE SYMÉTRIE

Si l'axe 3 est un axe de symétrie pour le core, on peut construire une infinité de trièdres K_α à partir de K_i. Tous ces trièdres se déduisent les uns des autres par des rotations finies d'angle ψ_0 autour de l'axe de symétrie. La fonction d'onde $\Psi^{\tau EIM}$ est alors invariante

vis-à-vis de l'application de l'opérateur $R_2^c(\psi_0)$ qui fait tourner le référentiel K_α de ψ_0 autour de l'axe 3. On a

$$\begin{aligned} &R_2^c(\psi_0)D^I_{MK} = e^{iK\psi_0}D^I_{MK}, \\ &R_2^c(\psi_0)\langle x_{a\alpha}|\eta J\Omega\rangle = e^{-i\Omega\psi_0}\langle x_{a\alpha}|\eta J\Omega\rangle. \end{aligned} \tag{26}$$

En outre, dans ce cas, $\gamma = 0$ tandis que I_3 et J_3 deviennent des constantes de mouvement. De (26) on déduit $K = \Omega$, d'où au lieu du développement (22),

$$\Psi^{\tau EIMK}_{\text{sym}} = N_{IK}(D^I_{MK}G^{\tau EI}_K + (-1)^{I+K}D^I_{M-K}R_1G^{\tau EI}_K) \tag{22'}$$

avec

$$G^{\tau EI}_K(x_{a\alpha}\beta\gamma) = \sum_{\eta J}\varphi^{\tau EI\eta J}_{KK}(\beta,\gamma)\langle x_{a\alpha}|\eta JK\rangle.$$

L'opérateur R_2 étant un cas particulier de l'opérateur R_2^c, les relations (25) restent valables.

5. L'équation de A. Bohr et ses conditions de validité

Afin de chercher de façon approchée les valeurs propres de H, A. Bohr [2]) fait les hypothèses suivantes, que nous énumérerons, pour des raisons de simplicité, en nous bornant au cas d'une seule particule couplée avec le core:

(i) le couplage core-nucléon est fort;
(ii) les opérateurs I_3, J_3, $\mathbf{J}^2$ sont de bons nombres quantiques;
(iii) les termes de H non diagonaux en I_3, J_3, J, peuvent être négligés en première approximation.

Avec ces hypothèses, on obtient un core à symétrie axiale et la fonction d'onde déduite de (22') est du type ($K > 0$)

$$\Psi^{\tau EIMK} = \sqrt{\frac{2I+1}{16\pi^2}}\varphi^{\tau EI\eta J}_{KK}(\beta,\gamma)\{D^I_{MK}\langle x_a|\eta JK\rangle + (-1)^{I-J}D^I_{M-K}\langle x_a|\eta J-K\rangle\}.$$

L'équation (6) devient alors †

$$\{E_\eta - E + T_{\text{vib}} + V + \langle\eta JK|H_{\text{int}}|\eta JK\rangle + \langle IMK\eta JK|T_{\text{rot}}|IMK\eta JK\rangle\}\varphi^{\tau EI\eta J}_{KK}(\beta,\gamma) = 0, \tag{25}$$

ou E_η est la valeur propre de H_p (dans ce cas particulier où un seul nucléon est couplé au core) correspondant à $\langle x_a|\eta JK\rangle$. L'équation (25) est exactement l'équation utilisée par A. Bohr et c'est le terme $\langle\ldots|T_{\text{rot}}|\ldots\rangle$ qui conduit au spectre bien connu des niveaux de rotation.

Il convient de remarquer que toutes les hypothèses faites pour

† L'équation (25) comporte des termes supplémentaires si $K = 1/2$.

obtenir l'équation (25) ne peuvent être retenues. Examinons en particulier si J_3 et $\mathbf{J}^2$ sont de bons nombres quantiques. Pour cela calculons $[J_3,H]$ et $[\mathbf{J}^2,H]$. On a d'abord comme crochet de Poisson différent de zéro,

$$[J_3, H_{\text{int}}] \sim K(r)\beta \sin\gamma \cdot n_1 n_2,$$

terme négligeable puisque $\gamma \approx 0$. Mais il n'en est pas de même pour

$$[\mathbf{J}^2, H_{\text{int}}] \sim K(r)\beta(2n_3^2 + n_1^2 - n_2^2),$$

qui n'est négligeable que si $K(r)$ ou β sont petits, *deux hypothèses qui impliquent un couplage faible*, en contradiction avec l'hypothèse (i). On peut toutefois éviter cette difficulté en remarquant qu'il n'est pas nécessaire de supposer que J_3 et $\mathbf{J}^2$ soient de bons nombres quantiques, ainsi qu'on le verra aux § 6 et 7.

6. Le spectre de rotation des noyaux asymétriques [6])

On remarquera que des deux termes de couplage core-nucléons l'un, T_{rot}, est en β^{-2} et l'autre, H_{int}, est en β. Dans le cas du couplage fort, où β est grand, il sera donc permis de traiter T_{rot} comme une perturbation, à condition toutefois que le core n'ait pas une forme symétrique, ce que nous admettrons (le cas de $\gamma = 0$ étant examiné au § 7).

L'hamiltonien $\mathscr{H}$ de l'approximation d'ordre zéro est

$$\begin{aligned} \mathscr{H} &= H - T_{\text{rot}} \\ &= H_p + T_{\text{vib}} + V + H_{\text{int}} \end{aligned} \tag{26}$$

et ne contient ni les angles d'Euler, ni leurs dérivées. Il en résulte que

$$[I_3, \mathscr{H}] = 0, \tag{27}$$

ce qui veut dire que l'approximation d'ordre zéro est caractérisée un nombre quantique supplémentaire $|K|$ (et non K en raison de (22)). Si $\mathscr{E}$ est valeur propre de $\mathscr{H}$ on a

$$(\mathscr{H} - \mathscr{E})\Psi^{\tau EIMK} = 0 \quad (K \geqq 0). \tag{28}$$

Les fonctions $G_K^{\tau EI}$ et $R_1 G_K^{\tau EI}$ qui figurent dans Ψ sont alors explicitement indépendantes de I et K, sauf peut-être par un facteur de phase. On les écrira donc $G^{\tau\mathscr{E}}$ et $R_1 G^{\tau\mathscr{E}}$ et elles sont solutions de

$$(\mathscr{H} - \mathscr{E})G^{\tau\mathscr{E}} = (\mathscr{H} - \mathscr{E})R_1 G^{\tau\mathscr{E}} = 0. \tag{29}$$

La fonction d'onde d'ordre 0 sera, d'après (22) et (29),

$$\psi^{\tau\mathscr{E}IMK} = \sqrt{\frac{2I+1}{16\pi^2}}\{D^I_{MK}G^{\tau\mathscr{E}} + (-)^{I+K}D^I_{M-K}R_1G^{\tau\mathscr{E}}\} \quad (K>0). \quad (30)$$

Si $K = 0$, la situation est un peu plus délicate, car pour que la forme générale (24') de $G_0^{\tau EI}$ ne dépende de I que par un facteur de phase, il faut que $I - J$ soit toujours pair ou toujours impair. L'éq. (30) devient alors

$$\psi^{\tau\mathscr{E}IM0} = \sqrt{\frac{2I+1}{8\pi^2}}D^I_{M0}G_0^{\tau\mathscr{E}}, \quad (30')$$

avec

$$G_0^{\tau\mathscr{E}} = \begin{cases} \sum\limits_{\substack{\eta J \\ \Omega\geq 0}} \varphi_{0\Omega}^{\tau\mathscr{E}\eta J}\{\langle x_{a\alpha}|\eta J\Omega\rangle + (1-\delta_{0\Omega})\langle x_{a\alpha}|\eta, J-\Omega\rangle\} & (I-J \text{ pair}) \\ \sum\limits_{\substack{\eta J \\ \Omega>0}} \varphi_{0\Omega}^{\tau\mathscr{E}\eta J}\{\langle x_{a\alpha}|\eta J\Omega\rangle - \langle x_{a\alpha}|\eta, J-\Omega\rangle\} & (I-J \text{ pair}). \end{cases} \quad (31)$$

Il en résulte que les valeurs de I compatibles avec $K = 0$ varient de 2 en 2. D'une façon générale, les fonctions $G^{\tau\mathscr{E}}$, $G_0^{\tau\mathscr{E}}$ dans leurs développements (24) et (31), comme on l'a vu plus haut, n'admettent que des valeurs de Ω variant de 2 en 2. *Il en résulte donc que les valeurs de K possibles, pour des nombres quantiques τ, $\mathscr{E}$, I, M donnés, varient elles-mêmes de deux en deux.*

Les énergies de rotation, $\mathscr{E}_{\text{rot}}$, s'obtiennent en diagonalisant la matrice $\langle\tau\mathscr{E}IMK'|T_{\text{rot}}|\tau\mathscr{E}IMK\rangle$. Posant

$$\mathscr{I}_\nu^{-1} = \frac{\hbar^2}{8B\beta^2}\,\frac{1}{\sin^2\left(\gamma - \frac{2\pi}{3}\nu\right)}$$

on pourra écrire T_{rot} sous la forme

$$T_{\text{rot}} = a(\beta,\gamma)(\mathbf{I}-\mathbf{J})^2 + b(\beta,\gamma)(I_3 - J_3)^2 + c(\beta,\gamma)\{(I_1-J_1)^2 - (I_2-J_2)^2\}, \quad (32)$$

avec

$$a = \tfrac{1}{2}\left(\frac{1}{\mathscr{I}_1} + \frac{1}{\mathscr{I}_2}\right), \quad b = \frac{1}{\mathscr{I}_3} - a, \quad c = \tfrac{1}{2}\left(\frac{1}{\mathscr{I}_1} - \frac{1}{\mathscr{I}_2}\right). \quad (33)$$

On trouve pour les éléments de matrice non nuls de T_{rot}:

a. *si* $K > 0$:

$$\left.\begin{aligned} \langle\tau\mathscr{E}IMK|T_{\text{rot}}|\tau\mathscr{E}IMK\rangle &= \mathscr{A}I(I+1) + \mathscr{B}K^2 + \mathscr{C}K + \mathscr{D} - (-1)^{I+\frac{1}{2}}\delta_{K,\frac{1}{2}}(I+\tfrac{1}{2})\mu \\ \langle\tau\mathscr{E}IM,K+2|T_{\text{rot}}|\tau\mathscr{E}IMK\rangle &= \frac{(c)}{2}\{f_I(-K-2)f_I(-K-1) + f_I(K)f_I(K+1)\}, \end{aligned}\right\} \quad (34)$$

où $f_I(K) = \sqrt{\{I(I+1) - K(K+1)\}}$ et $\mathscr{A}, \mathscr{B}, \ldots, \mu, (c)$ sont des coefficients caractéristiques de l'approximation d'ordre zéro, construits à partir d'intégrales des types suivants:

$$(u) = \int G^{\tau\mathscr{E}*} U G^{\tau\mathscr{E}}\, dv, \quad (u)' = \int (R_1 G^{\tau\mathscr{E}})^* U (R_1 G^{\tau\mathscr{E}}) dv,$$

$$(u)_m = \int (R_1 G^{\tau\mathscr{E}})^* U G^{\tau\mathscr{E}}\, dv,$$

où U est un opérateur quelconque $U(\beta, \gamma, x_{\alpha\alpha}, \ldots)$. On a

$$\mathscr{A} = \tfrac{1}{2}\left[(a) + (a)' + (-1)^{I+1}\delta_{K1}\,\frac{(c)_m + (c)_m^*}{2}\right],$$

$$\mathscr{B} = \tfrac{1}{2}[(b) + (b)'],$$

$$\mathscr{C} = -([a+b]J_3) - ([a+b]J_3)',$$

$$\mathscr{D} = \tfrac{1}{2}[(a\mathbf{J}^2 + bJ_3^2 + c(J_1^2 - J_2^2)) + \{a\mathbf{J}^2 + bJ_3^2 + c(J_1^2 - J_2^2)\}'],$$

$$\mu = \tfrac{1}{2}[(a[J_1 - iJ_2])_m + (c[J_1 + iJ_2])_m + \text{compl. conj.}].$$

Les valeurs possibles de K, pour I donné, sont

$$\begin{aligned} &K = I, I-2, \ldots \\ &\qquad \text{ou bien} \qquad (K > 0). \\ &K = I-1, I-3, \ldots \end{aligned} \tag{35}$$

b. *si une des valeurs de K est nulle*: les fonctions $G^{\tau\mathscr{E}}$ étant alors du type (31) il en résulte que

$$\langle \tau\mathscr{E}IMK | T_{\text{rot}} | \tau\mathscr{E}IMK\rangle = \mathscr{A}_0 I(I+1) + \mathscr{B}_0 K^2 + \mathscr{D}_0$$

$$\langle \tau\mathscr{E}IM, K+2 | T_{\text{rot}} | \tau\mathscr{E}IMK\rangle \tag{34'}$$

$$= \frac{(c_I)}{2\sqrt{2}}\{f_I(-K-2) f_I(-K-1) + f_I(K) f_I(K+1)\},$$

avec

$$\mathscr{A}_0 = \frac{(a)}{2} + \delta_{K,1}(-1)^{I+1}\frac{(c)}{2}, \quad \mathscr{B}_0 = \frac{(b)}{2}, \quad \mathscr{D}_0 = (a\mathbf{J}^2 + bJ_3^2 + c(J_1^2 - J_2^2));$$

$$\begin{aligned} I &= 0, 2, 4, \ldots \quad \text{ou } I = 1, 3, 5, \ldots \\ K &= 0, 2, 4, \ldots \quad (0 \leqq K \leqq I). \end{aligned} \tag{35'}$$

7. Spectre de rotation des noyaux symétriques ou presque

Si le core a une forme voisine de celle d'un ellipsoïde de symétrie axiale, on a $\gamma \approx 0$. Il en résulte que dans T_{rot} le terme $1/\mathscr{I}_3$ est très grand et ne peut être traité comme une perturbation. Toutefois en incluant le terme $(I_3 - J_3)^2/\mathscr{I}_3$ dans l'hamiltonien d'ordre zéro, on

peut reprendre le raisonnement du § 6. L'hamiltonien d'ordre zéro sera

$$\mathscr{H}_{\text{sym}} = H_p + T_{\text{vib}} + V + H_{\text{int}} + \frac{(I_3 - J_3)^2}{\mathscr{J}_3}$$

et on aura toujours $|K|$ comme constante de mouvement. Les solutions de

$$(\mathscr{H}_{\text{sym}} - \mathscr{E})\Psi^{\tau\mathscr{E}IMK} = 0 \tag{36}$$

seront encore de la forme (30), mais les fonctions G dépendront explicitement de K:

$$\Psi^{\tau\mathscr{E}IMK} = \sqrt{\frac{2I+1}{16\pi^2}}\{D^I_{MK} G^{\tau\mathscr{E}}_K + (-1)^{I+K} D^I_{M-K} R_1 G^{\tau\mathscr{E}}_K\} \quad (K>0). \tag{30'}$$

Si $K = 0$, la fonction d'onde sera de la forme (30') avec

$$G^{\tau\mathscr{E}}_0 = \sum_{\eta J} \varphi^{\tau\mathscr{E}\eta J}_{00} \langle x_{\alpha\alpha} | \eta J \Omega \rangle;$$

on utilise le fait qu'à la limite $\gamma = 0$, J_3 est aussi une constante de mouvement et $K = \Omega$.

L'expression (32) du terme perturbateur à utiliser sera

$$T^{(\text{sym})}_{\text{rot}} = \frac{\hbar^2}{6B\beta^2}\{(\mathbf{I} - \mathbf{J})^2 - (I_3 - J_3)^2\}. \tag{38}$$

On retrouve alors le spectre de rotation bien connu de Bohr-Mottelson [1]):

$$\mathscr{E}^{(\text{sym})}_{\text{rot}} = \mathscr{A}_{\text{sym}} I(I+1) + \mathscr{D}_{\text{sym}} - \mathscr{A}_{\text{sym}} K^2 - (-1)^{I+\frac{1}{2}} \delta_{K,\frac{1}{2}} (I+\tfrac{1}{2}) \mu_{\text{sym}}. \tag{39}$$

Le seul nombre quantique qui puisse varier dans une même famille d'états de rotation est I, qui pour une valeur donnée de K est égal à

$$\begin{cases} I = K, K+1, K+2, \ldots & \text{si } K > 0 \\ I = 0, 2, 4, 6, \ldots & \text{si } K = 0. \end{cases} \tag{40}$$

Les coefficients $\mathscr{A}_{\text{sym}}$, $\mathscr{D}_{\text{sym}}$, μ_{sym} sont avec les notations du § 6

$$\mathscr{A}_{\text{sym}} = \frac{\hbar^2}{6B}(\beta^{-2}),$$

$$\mathscr{D}_{\text{sym}} = \frac{\hbar^2}{12B}\left[\left(\frac{\mathbf{J}^2 - J_3^2}{\beta^2}\right) + \left(\frac{\mathbf{J}^2 - J_3^2}{\beta^2}\right)'\right],$$

$$\mu_{\text{sym}} = \frac{\hbar^2}{12B}\left[\left(\frac{J - iJ_2}{\beta^2}\right)_m + \text{compl. conj.}\right].$$

8. Conclusion et résumé

En prenant le modèle unifié de A. Bohr pour base de discussion, les méthodes développées aux § 6 et 7 montrent que les spectres de rotations nucléaires existent sous des conditions très générales. L'hypothèse essentielle est celle d'un couplage fort entre le core et les N nucléons, mais en dehors de cela, il n'y a rien à supposer sur ces nucléons. En particulier, ils peuvent être représentés par des mélanges de configurations très généraux, ce qui permet de lever une difficulté essentielle du raisonnement de A. Bohr (§ 5).

Le modèle unifié conduit à distinguer deux types de spectres de rotation nucléaire:

a. si le core est asymétrique, les termes d'une même famille spectrale sont obtenus en diagonalisant la matrice T_{rot} (34) ou (34'), compte tenu des valeurs admissibles pour I et K (cf. (35) ou (35')). Ces deux degrés de liberté nécessitent des énergies du même ordre de grandeur pour être excités.

b. si le core a un axe de symétrie (ou s'il est voisin de cette forme), le degré de liberté K devient difficile à exciter par rapport à I. Les termes d'une même famille sont caractérisés par le nombre quantique I dont les valeurs possibles sont données en (40); les niveaux d'énergie correspondants sont de la forme (39).

Les données expérimentales apportent, pour les noyaux très déformés, une bonne confirmation de l'existence de spectres de rotation du type symétrique [1]). Il est ainsi permis de conclure que la plus grande partie des noyaux déformés sont à symétrie axiale. Toutefois, une étude plus détaillée sera reprise à ce sujet.

L'auteur tient à remercier les Prof. Joliot, Proca et Rosenfeld, pour l'intéret qu'ils ont porté à ce travail, ainsi que MM. A. Bohr et Nataf, pour d'utiles critiques du manuscrit initial. Ce travail a été possible, en outre, grâce à une allocation du Centre National de la Recherche Scientifique.

Appendice

FORME DE L'HAMILTONIEN DU SYSTÈME DANS LE REPÈRE MOBILE K_α

Il s'agit d'exprimer l'hamiltonien(5)

$$H = H_p + H_e + H_{\text{int}}$$

à l'aide des degrés de liberté du système mobile K_α connaissant H dans le système K_i.

Les variables indépendantes sont

dans K_i: $x_{ai}, s_{ai}, \psi_{ij}$; $p_{ai} = -i\hbar \dfrac{\partial}{\partial x_{ai}}$, $\pi_{ij} = -i\hbar(1+\delta_{ij}) \dfrac{\partial}{\partial \psi_{ij}}$;

dans K_α: $x_{a\alpha}, s_{a\alpha}, q_\mu, \theta_u$; $p_{a\alpha} = -i\hbar \dfrac{\partial}{\partial x_{a\alpha}}$, $p_\mu = -i\hbar \dfrac{\partial}{\partial q_\mu}$, $p_u = -i\hbar \dfrac{\partial}{\partial \theta_u}$.

On a, bien évidemment,

$$\begin{aligned} x_{ai} &= \sum_\alpha S_{i\alpha}(\theta) x_{a\alpha} \\ \psi_{ij} &= \sum_\mu q_\mu S_{i\mu}(\theta) S_{j\mu}(\theta). \end{aligned} \tag{A1}$$

En raison de la présence des opérateurs de spin s_{ai}, on doit en dehors des substitutions $(A1)$ effectuer une transformation unitaire $U(\theta)$ liant les variables indépendantes q_i d'un référentiel K_i à celles, q_α, du référentiel K_α:

$$q_i = U(\theta) q_\alpha U^\dagger(\theta) \tag{A2}$$

Dans le cas où une seule particule est couplée au core, la matrice U se réduit à la matrice $D^{\frac{1}{2}}$ qui lie les fonctions de base de la représentation d'ordre $\frac{1}{2}$ du système K_i à celles du système K_α (cf. § 4). Dans le cas des opérateurs de spin S_{ai}, la transformation canonique $(A2)$ est équivalente à

$$s_{ai} = \sum_\alpha S_{i\alpha}(\theta) s_{a\alpha}.$$

Pour la transformation des moments conjugués, on utilise les relations différentielles

$$U p_\mu U^\dagger = \tfrac{1}{2} \sum_{ij} \frac{\partial \psi_{ij}}{\partial q_\mu} \pi_{ij}, \tag{A3}$$

$$U p_u U^\dagger = \tfrac{1}{2} \sum_{ij} \frac{\partial \psi_{ij}}{\partial \theta_u} \pi_{ij} + \sum_{ai} \frac{\partial x_{ai}}{\partial \theta_u} p_{ai}, \tag{A4}$$

$$U p_{a\alpha} U^\dagger = \sum_i \frac{\partial x_{ai}}{\partial x_{a\alpha}} p_{ai}. \tag{A5}$$

On remarque que les matrices U disparaissent de (A3, 5) et que (A5) s'inverse tout de suite en donnant

$$p_{ai} = \sum_\alpha S_{i\alpha}(\theta) p_{a\alpha}. \tag{A6}$$

Suivant Wentzel [3]), au lieu d'employer (A4) on peut se servir d'une relation équivalente où les p_u sont remplacés par les trois opérateurs

différentiels I_α de moment angulaire total. A la place de (A4) on a

$$\hbar U I_\alpha U^\dagger = \tfrac{1}{2} \sum_{ir} q_\mu (I_\alpha S_{i\mu} S_{j\mu}) \pi_{ij} + \frac{i\hbar}{2} \sum_{i\beta\gamma} S_{i\beta} (I_\alpha S_{i\gamma}) L_{\gamma\wedge\beta}. \quad \text{(A7)}$$

Dans l'équation (A7) on a fait apparaître l'opérateur L_α du moment orbital total des nucléons exprimé dans les axes mobiles. L'opérateur différentiel I_α ne commute pas avec U, mais on a $I_\alpha U^\dagger = U^\dagger I_\alpha - U^\dagger S_\alpha$, où S_α est l'opérateur de spin total des N nucléons. Le système à résoudre devient, après quelques transformations,

$$\begin{cases} p_\mu = \tfrac{1}{2} \sum_{ij} \dfrac{\partial \psi_{ij}}{\partial q_\mu} \pi_{ij} \\ \hbar (I_{\alpha\wedge\beta} - J_{\alpha\wedge\beta}) = \tfrac{1}{2} \sum_{ij} (q_\alpha - q_\beta)(S_{i\alpha} S_{j\beta} + S_{i\beta} S_{j\alpha})\, \pi_{ij}. \end{cases}$$

On vérifie sans peine que la solution de ce système linéaire est π_{ij}

$$\pi_{ij} = 2 \sum_{\mu} S_{i\mu} S_{j\mu} p_\mu + \frac{\hbar}{2} \sum_{\alpha\beta} (S_{i\alpha} S_{j\beta} + S_{i\beta} S_{j\alpha}) \frac{I_{\alpha\wedge\beta} - J_{\alpha\wedge\beta}}{q_\alpha - q_\beta}. \quad \text{(A8)}$$

Jointes à (12), les expressions (A1, 6, 8) permettent de calculer l'hamiltonien H en fonction des nouvelles variables.

Par comparaison avec les résultats de Bohr [2]) et Wentzel [3]), on notera que H a la même expression que celle qu'on aurait obtenue en faisant un changement de variables sur H_p et H_c séparément, mais *à condition* que dans le terme H_c ainsi obtenu, l'énergie de rotation $\Sigma_\nu [Q_\nu^2 / \sin^2(\gamma - \frac{2}{3}\pi\nu)]$ soit exprimée à l'aide des variables indépendantes I_ν et J_ν conformément à (8'').

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Nuclear Physics
1 (1956)

THE JOINING OF INFRA-RED AND ULTRA-VIOLET CALCULATIONS

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Received 3 May 1955

Abstract: The perturbation radiation corrections to electron scattering diverge as the energy resolution ΔE vanishes [1]). This makes terms, of order higher than the first power of the fine structure constant α, important for small values of ΔE already obtained experimentally. The Bloch-Nordsieck type calculations [2,3,4]) show that the high order, low energy photon processes are essential, and that the above mentioned divergence is due to the inadequacy of the perturbation method for these processes. However, this type of calculation is invalid for high energy virtual photons. Using the recent formulation of the Bloch-Nordsieck method [4]), and a perturbation treatment cut off for small photon energies, an expression for certain events is found, correct to any chosen order in α and to first order in a parameter of magnitude $< 10^{-2}$.

1. Introduction

The problem of the infra-red divergence has been solved in a certain sense. The perturbation radiation corrections, for example of purely elastic scattering from a weak potential [1]), contain a logarithmic divergence for vanishing virtual photon energy. But in an actual experiment, the inelastic events in which the electron energy changes by an amount less than the experimental energy resolution ΔE, cannot be distinguished from the elastic events. These events have a contribution to the matrix element proportional to ${}_0\int^{\Delta E} dw/w$ and the contribution from the lower limit exactly cancels the infra-red divergence due to the virtual photons in every order [4]). However the upper limit contributes terms proportional to $\ln \Delta E$, and it is the effect of these terms with which we are concerned. As ΔE decreases, the radiation corrections converge more slowly (in whatever sense the perturbation series converges at all) eventually making the higher order terms larger than the zeroth order expression, with ultimate divergence in the limit $\Delta E \to 0$.

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The first order expression gives rise to negative cross sections for small ΔE. This effect of small ΔE on the convergence of the perturbation expansions is often dismissed [2]) on the ground that if $-\alpha \ln (\Delta E/mc^2) = 1$ then $\Delta E/mc^2 = \exp(-1/\alpha)$, which is far beyond present experimental technique. However a more detailed examination of the second order term in electron scattering from a weak external potential indicates the error of such an argument. In this case, for relativistic electron energies, the part of the first order term proportional to $\ln \Delta E$ is given by

$$-(4\alpha/\pi) \ln (\Delta E/E)[\ln (2E/mc^2) \sin \tfrac{1}{2}\theta - \tfrac{1}{2}],$$

where E is the electron total energy, and θ the scattering angle in the centre of mass system. For large angle scattering, the term in brackets is large. We further note that it may be pertinent to enquire when this term is equal to some fraction f, as the first order correction will certainly be inadequate when it is a large fraction of the zeroth order term.

Thus our criterion is $\Delta E/mc^2 = E/mc^2 \exp(-1/\alpha g)$, where $g = (4/f\pi) \ln (2E/mc^2)$. As g appears in the exponent its effect on $\Delta E/mc^2$ is very great, making the critical value of ΔE much larger than the previous criterion. In § 3 we shall discuss applications, and will see that the disturbing effects of $\ln \Delta E$ are already present in the analysis of certain experiments †.

While the usual perturbation series breaks down as $\Delta E \to 0$, the physics of the situation for soft photons was clarified by Bloch and Nordsieck [2]). They noted that, when the photon wavelength was large compared to the compton wavelength, the electron current could be treated as an external current and the radiation field obtained exactly. They could then show, and this is confirmed by the more adequate calculation of Jauch and Rohrlich [2]), that an infinite number of low energy photons are emitted, whose total energy is however finite. The resultant effect on the scattering vanishes for large ΔE. For $\Delta E \to 0$, the scattering vanishes, corresponding to the fact that a charged particle radiates when it is accelerated. This indicates clearly that the defect in the perturbation calculation is due to the fact that the soft photon processes are essentially of very high order.

† Independently of the size of $\ln \Delta E$, the perturbation series, as is well known, becomes invalid for very large E (due to terms proportional to $\ln E$). However, there is a large range of E, for which terms proportional to $\ln \Delta E$ are large for reasonable ΔE, while terms proportional only to $\ln E$ are still small.

The problem that remains is to join the Bloch-Nordsieck method for soft photons to the perturbation method for hard photons. Using Jauch and Rohrlich's [4]) formulation of the Bloch-Nordsieck method this becomes a simple task. Their correction factor $b(E, \Delta E, \varepsilon)$ for the soft photon effects of high energy electron scattering from a weak potential, is taken with a straight high energy cut-off at $\varepsilon(E \exp(-1/\alpha g) < \varepsilon < E)$. This is then multiplied by the perturbation expression to a given order in α with a low energy cut off at ε; that is, ε replaces ΔE in the usual perturbation series† and this new series is multiplied by b. The errors (and amount of cut off dependence involved) are of the next order in α and $O[\varepsilon/(E - mc^2)]$. This latter error can be made small in the cases to be discussed, while the convergence of the perturbation series is kept sufficiently rapid.

As $\Delta E \to 0, b \to 0$; thus, this new expression has the proper behaviour in this limit. For $\Delta E/E > \exp(-1/\alpha g)$ one can expand b in powers of α. Multiplying this by the perturbation series to a given order in α, one obtains just the usual perturbation series to that order, the ε dependence vanishing to $O[\varepsilon/(E-mc^2)]$.

In § 2 we shall present the formalism of the above "joining". In § 3 we shall apply this to recent scattering experiments. § 4 contains a discussion of the generalization of these methods (1) for strong (or long range) interactions, (2) for processes other than scattering and (3) for scattering of low energy electrons.

2. Formalism

We must first discuss the results of Jauch and Rohrlich [4])†† We are in disagreement with the statement in I of the approximations involved in obtaining their result. As the application of their result is of prime interest in this discussion, it will be necessary to treat its range of validity with care.

To obtain the result of I it is necessary to separate the soft photon radiation effects from the other effects being considered, so that the infra-red contribution may be written as a factor to the matrix element for the other processes. In this separation an error is in-

† It is to be noted that when the contributions for $0 < \omega < \Delta E$ are $O(1)$ or $O(\Delta E)$ the calculations in the literature [1, 5]) do not cut the integrals off at ΔE, as they do for the terms $O(\ln \Delta E)$. For the terms of $O(\Delta E)$ this is unimportant, but in the terms $O(1)$ the error should be corrected before comparison with experiment. In our formalism, ε replaces ΔE also in these terms

†† This paper will hereafter be referred to as I.

volved and it is our contention that this error is $O[\varepsilon/(E-m)]$ ($\hbar = c = 1$; ε is the maximum energy of the soft photons), not, as is stated in I, $O(\varepsilon/m)$.

This separation is explicitly done in I only for their perturbation calculation of the infra-red effects. For example, first order real photon emission is represented by the following two diagrams (cf. fig. 1),

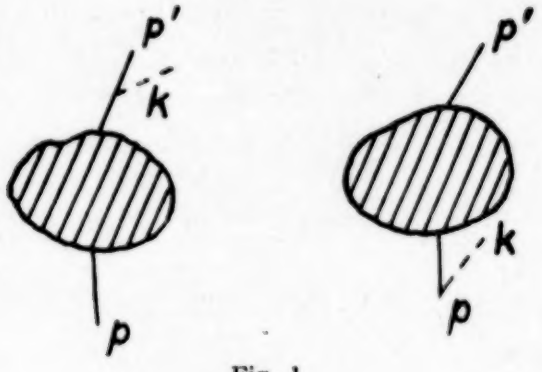

Fig. 1.

where the shaded area represents other processes (including the emission of real photons) with matrix element Q. The matrix element for the above diagrams is given by I (2), which is then approximated by I (3). This clearly assumes that e.g. $Q(p', p-k) = Q(p', p) + O(\omega/m)$, or that the soft photon processes affect the processes in the shaded area to $O(\omega/m)$. However, this is not true in many important cases; e.g. the zeroth and first order matrix elements for scattering from an external potential, which are sensitive functions of the kinetic energy of the electron. Thus

$$Q(p', p-k) = Q(p', p) + O[\omega/(E-m)].$$

Now, this approximation is sufficient for the derivation of b. For if we have $\omega \ll E' - m$ and use $\bar{u}(p')\,(iP' + m) = 0$†, with $p'^2 = -m^2$, as well as $\mathbf{k}\cdot\mathbf{e} = 0$, $k^2 = 0$, we get

$$\bar{u}(p')\bar{E}\left(\frac{i(P'+K)-m}{(p'+k)^2+m^2}\right) = \bar{u}(p')\frac{-i(P'+K)\bar{E}-m\bar{E}+2i\mathbf{p}'\cdot\mathbf{e}+2i\mathbf{k}\cdot\mathbf{e}}{2p'k}$$

$$= \frac{i\bar{u}(p')\,(\mathbf{p}'\cdot\mathbf{e} - \tfrac{1}{2}K\bar{E})}{\omega(|\mathbf{p}'|\cos\theta' - E')} = i\,\frac{\bar{u}(p')\mathbf{e}\cdot\mathbf{p}'}{\omega(|\mathbf{p}'|\cos\theta' - E')} \qquad (1)$$

and a similar expression for the term involving p, thus giving I (4), and the necessary separation is effected.

† $P = p_\mu\gamma_\mu$, $K = k_\mu\gamma_\mu$, $\bar{E} = e_\mu\gamma_\mu$; e_μ is the polarization vector.

This same approximation is also relevant to the closed form of the soft photon processes. Here Jauch and Rohrlich replace the matrix element corresponding to the diagram of fig. 2, in which any number of soft photon processes (real or virtual) can occur on the external photon lines, by the product of the matrix elements for the shaded area and for the classical current distribution of a point electron moving with four velocity p_μ/m for $x_0 < 0$ and p'_μ/m for $x_0 > 0$. A requirement here is obviously that the matrix element for the shaded area be independent of the soft photon processes or the external lines, or again that $\omega \ll E - m$. Bloch and Nordsieck have previously noted this requirement.

Fig. 2.

These latter authors also state two other criteria for the validity of their method:

(a) $1/\omega >$ classical electron radius,

(b) $1/\omega >$ electron compton wavelength.

The explicit formalism of I does not require (a) and (b) [providing we are correct in substituting $\omega \ll E - m$ for (b)] but involves implicit assumptions which may require one or both when the perturbation theory is not valid for the high energy processes. One assumption is that the "other processes", indicated by the shaded part of the graph, are not so strong or of such long range that the substituted classical current distribution does not approximate the electron path as seen by the soft photons. In the case that these processes are due to the zeroth order interaction (in the radiation field) they can be easily included in the formalism, by using the external current appropriate to this interaction, as we shall discuss in § 4. The hard radiation interaction also disturbs the current by causing fluctuations over distances of the order of the electron compton wavelength. If this latter interaction is strong, the electron current will nowhere be the external current assumed. But if the soft photon wavelength satisfied (b), then it would see only an average aspect of the electron path, and the approximation would certainly be correct.

However, if perturbation theory is valid for hard photons, one may consider an alteration of the prescribed external current due to the emission and absorption of a hard photon of energy $\omega > \varepsilon$. Then instead of an electron of velocity $v_0 = (E^2 - m^2)^{\frac{1}{2}}/E$, one must consider one of velocity $v_0 - \Delta v(t)$, with

$$\Delta v(t) = \begin{cases} 0 & -\infty < t < -\frac{a}{2} \\ \frac{\sqrt{E^2 - m^2}}{E} - \frac{\sqrt{(E-\omega)^2 - m^2}}{E-\omega} & -\frac{a}{2} < t < \frac{a}{2} \\ 0 & \frac{a}{2} < t < \infty \end{cases}$$

where $a = 1/\omega$ (as $\Delta E \Delta t = \omega \Delta t = 1$).

By fourier analyzing $\Delta v(t)$ we can then apply a formula of Heitler [6]) to find the soft radiation from each component. Integrating over components one can then easily compare the radiation obtained to that of the unaltered electron distribution. The result is that $\Delta A_k/A_k \leqq O\left(\varepsilon/(E-m)\right)$ where A_k is the strength of field component of momentum $k < \varepsilon$ and ΔA_k is the variation of this quantity due to the temporary change of velocity of the electron. This result can be understood by the observation that the long wave radiation is affected by variations of the electron velocity by an amount proportional to the change in velocity Δv and to the ratio of the interval of disturbance to the period of field component concerned, k/ω. When $\omega \ll E - m$, then $\Delta v = \omega^2/E^2$, and the effect is $O\big((k/E) \cdot (\omega/E)\big) < O\big(\varepsilon/(E-m)\big)$. When $\omega \approx E - m$, then $\Delta v \approx \sqrt{E^2 - m^2}/E < 1$, and the effect is $< O(k/(E-m))$ i.e. $< O\big(\varepsilon/(E-m)\big)$.

It is here that the assumption of the convergence of the perturbation theory, for hard radiation corrections, must be made. If the convergence is only asymptotic, then the following development, which includes the use of the perturbation series for hard photons, is only valid when the order of the series is suitably small. With the assumption of convergence the process of emitting and absorbing a single hard photon yields an error in the soft photon field of $O\big(\alpha\, \varepsilon/(E-m)\big)$, while the process for n photons $(a \approx n/\omega)$ yields an error of $O\left(\alpha^n\, n\varepsilon/(E-m)\right)$. The total error is thus of $O\big(\sum_{n=1}^{\infty} n\alpha^n\, \varepsilon/(E-m)\big) \approx O\big(\alpha\, \varepsilon/(E-m)\big)$. Thus this error is always less by $O(\alpha)$ than the one due to the approximation involved in handling $Q(p'-k, p)$ and in deriving eq. (1).

The other assumption implicit in the work of I is that the polarization of the vacuum due to the soft photon field is negligible. Assuming convergence of the perturbation series for non infra-red divergent processes the order of the effect is bounded by the first

order pair creation and annihilation by a field component of wave number $k < \varepsilon$. This is $O(\ln [1 + (k^2/4m^2)]) < O(\alpha\, k/m) < O(\alpha\, \varepsilon/m)$. There is of course a possible dependence on the energy of the external current through its effect on A_k. But when $\alpha\, \varepsilon/m > \varepsilon/(E-m)$ (this is the only case in which we need consider this correction in addition to those of $O(\varepsilon/(E-m))$, then $E - m > m/\alpha$. This is then highly relativistic and the electron velocity has negligible energy dependence. The A_k are determined only by the electron velocity (as we are treating an external current there are no energy dependent recoil effects) and thus in this case we get no reduction of the order of the effect depending on $m/(E-m)$ and the error due to vacuum polarization from the soft field is greater than our previous errors, when $E - m > m/\alpha$.

However, as the vacuum polarization of an external electromagnetic field can be calculated in perturbation theory, without infrared divergence difficulties, we can take these effects into account when $E - m > m/\alpha$.

Thus if the renormalized perturbation expansion converges (for suitably chosen ε) we may drop (a) and (b), for it is only the renormalized theory that is relevant. This follows from the fictitious nature of the graphs which give rise to renormalization effects; that is, because one can simply remove contributions from those graphs [7]). There is no renormalization for b. For if we put (see eq. (3)) $p' = p$, then $\beta' = 0$, and thus $C = 0$. Then $b = 1$ as it should for the free electron.

Keeping in mind the above conditions we may now use the result of Jauch and Rohrlich, which is that

$$\sigma(\Delta E, p', p) = b(\Delta E, \varepsilon, p', p)\, \sigma_H(p', p) + O(\varepsilon/(E-m)) + O(\varepsilon/(E'-m)); \qquad (2)$$

$\sigma(\Delta E, p'\, p)$ is the cross section for an incoming electron with four-momentum p to come out with four momentum between $(\mathbf{p}', E')$ and $(\mathbf{p}', E' - \Delta E)$; σ_H is the cross section for the non infra-red divergent processes; ε is the cut off energy beyond which the radiation corrections are not included in b. For a weak interaction, with a straight cut-off at ε (see I eq. (56))

$$b(\Delta E) = \frac{1}{2\pi} \int_0^{\Delta E} dx \int_{-\infty}^{+\infty} d\sigma\, \exp\left\{ -ix\sigma + \alpha\, C \int_0^{\varepsilon} \frac{dw}{w} (e^{iw\sigma} - 1) \right\} \qquad (3)$$

where

$$C = \frac{2}{\pi}\left(\frac{\tanh^{-1}\beta'}{\beta'} - 1\right), \quad \beta' = \frac{|\mathbf{p}'|^2}{E'} \text{(in the frame in which } |\mathbf{p}| = 0).$$

The only requirement now is to properly define σ_H so that σ corresponds to the cross section predicted by quantum electrodynamics. It is evident that we have the complete expression for σ if σ_H is the expression for the zeroth order scattering, plus all radiative corrections due to photons with $\omega > \varepsilon$, and, when $E - m > m/\alpha$, the effect of the vacuum polarization due to the soft field.

In actual practice only the perturbation expression of σ_H up to a given order n can be computed:

$$\sigma_H(p, p') = \sigma^{(n)}\ (\varepsilon, p, p') + O\big((\alpha \ln \varepsilon)^{n+1}\big).$$

As ε is not now a vanishingly small number, the last term is actually $O\ (\alpha^{n+1})$. Thus,

$$\sigma(\Delta E, p, p') = b(\Delta E, \varepsilon, p, p')\ \sigma^{(n)}(\varepsilon, p, p') + O\big((\alpha \ln \varepsilon)^{n+1}\big) + O(\varepsilon/(E-m)) + O(\varepsilon/(E'-m)). \quad (4)$$

We note that the possible accuracy increases with n. For if n is larger we are free to increase $\ln \varepsilon$, and thus to decrease ε.

For $\Delta E = 0$, $b = 0$, and thus $\sigma = 0$ as required.

For $\Delta E/E \gg \exp(-1/\alpha g)$ we can expand b to nth order, in which case

$$\sigma = b^{(n)}\sigma^{(n)}(\varepsilon) + O(\alpha^{n+1}) + O(\varepsilon/(E-m)) + O(\varepsilon/(E'-m)) = \sigma^{(n)}(\Delta E) + O(\alpha^{n+1}), \quad (5)$$

$$b^{(n)}\sigma^{(n)}(\varepsilon) = \sigma^{(n)}(\Delta E) + O\,(\alpha^{n+1}) + O(\varepsilon/(E-m)) + O(\varepsilon/(E'-m)), \quad (6)$$

where $\sigma^{(n)}\ (\Delta E)$ is just the usual nth order perturbation expression. For instance, the Schwinger[1]) relativistic calculation for the first order radiation correction to scattering from a weak central potential is given by†

$$\sigma^{(1)}(\varepsilon, p, p') = \sigma^{(0)}\ (p, p')\ (1 - \delta\ (\varepsilon, p, p')) \quad (7)$$

where

$$\delta(\varepsilon) = \alpha C \ln\ (E/\varepsilon) + \alpha A(p, p') + O(\varepsilon/(E-m)); \quad (8)$$

† While Schwinger [1]) cuts off the logarithmically divergent term at low photon frequencies, he puts the cut off to zero in all other terms. In the first order expression all these terms differ from the properly cut off terms by $O(\varepsilon/(E-m))$. There is of course the possibility of differences of $O(1)$ arising. This in fact occurs in the second order calculations (see [5])) where however there remain no logarithmically diverging terms.

also

$$b^{(1)} = 1 - \alpha C \ln (\varepsilon/\Delta E). \tag{9}$$

Thus

$$b^{(1)}\sigma^{(1)} = \sigma^{(0)}(p, p')\,[1 - \alpha C(\ln (E/\varepsilon) + \ln (\varepsilon/\Delta E)) + \alpha A (p', p)] + O(\alpha^2) + O(\varepsilon/(E-m)) + O(\varepsilon(E'-m)). \tag{10}$$

As $\ln (E/\varepsilon) + \ln (\varepsilon/\Delta E) = \ln (E/\Delta E)$ we see that the correct expression is obtained for the ordinary perturbation expression cut off at ΔE†.

In the next section we will show the possibility of choosing ε so that the errors are small in experimental situations of present interest; in fact we show that a considerable increase in accuracy over that of the usual perturbation theory is possible for cases within reach of experiment.

3. Application to High Energy Scattering Experiments

In the usual perturbation expansion the error involved in the n th order expression increases as ΔE is decreased as $(\ln \Delta E)^{n+1}$. The error in eq. (4) does not depend on ΔE, but only on ε. Thus for ΔE less than a certain value (depending on E), the eq. (4) will have less error than the usual perturbation expansion to the same order n, for a suitably chosen ε.

In recent high energy electron scattering experiments at Stanford [8]) at 125 MeV, the beam energy spread was $\approx 1/4$ %. If such an experiment is done for the purpose of measuring the infra-red contributions (energy resolution of the detector comparable to the beam width; and preferably on a proton target to eliminate finite size and nuclear level effects) the first order radiative term $\delta(\Delta E) \approx \frac{1}{4}$ at large angles. The error involved in neglecting higher order terms is thus (assuming the absence of anomalously large coefficients) $\approx 1/16$. If in eq. (4) we use $\varepsilon/(E-m) \approx 1$ %, then $\delta(\varepsilon) \approx \frac{1}{5}$ and the estimated error is $\approx 1/25$. Adding the error of $O(\varepsilon/(E-m))$ we see that for ΔE as obtained in these experiments our expression has the same order of magnitude error as the usual perturbation expression. However, as ΔE decreases the error in equation (4) does not change; but if the energy resolution were improved by a

† Vacuum polarization from the soft field does not enter until the next order, when b is expandable; for in that case the emission and absorption of the soft photon is itself of order α.

factor of ten, then $\delta(\Delta E) \approx 2/5$ and the estimated error in the usual perturbation formula would be of the order of 4/25 if the series converges at all for such ΔE.

In scattering experiments at 5 MeV, the very good energy resolution of a Van de Graaf may possibly be used to obtain a similar discrepancy between $\sigma^{(1)}(\Delta E)$ and $b^{(1)}\sigma^{(1)}(\varepsilon)$. The experimental verification of our result is therefore feasible.

4. Generalization

The modification required to treat the case of a strong external potential such as the coulomb field is simply the substitution of the zeroth order current appropriate to that potential for S_μ. Thus the current would have the correct form when the effects of the hard photon field can be neglected.

At low energies, when the Born approximation for the coulomb field is no longer good, one would use [9]) the coulomb functions in computing S_μ. One can then obtain b (at least numerically), with great simplicity compared to computing the higher order matrix elements with coulomb functions.

A similar modification can be made for processes in which the number of external electron lines is not two. Again, one need only employ the proper S_μ†.

The problem of treating the infra-red field for slow electrons is complicated by the dependence of the matrix element for the "other processes" on the soft photon momenta. This requires the use of an expression $M = \int \beta(k)\, Q(p-k)\, dk$ where $\beta(k)$ is the matrix element for the soft radiation component with total momentum k. Progress is being made along these lines.

5. Conclusions

A method has been developed for treating the effects of the soft photon field to all orders, and simultaneously the effect of the hard photon field to a given order. It is seen that the method is useful for avoiding high order perturbation calculations, when the important parts of these terms are due to the soft photon field. A fortiori, one can avoid calculation of these higher order terms when

† Besides the formulation in I, there is the work contained and referred to in [10]).

they are complicated by the insufficiency of the Born approximation for strong external fields.

The author wishes to express his gratitude to Professor C. Møller for his kind interest. He is greatly indebted to G. Källén for many discussions and criticisms of the contents of this paper, and to A. Visconti for discussions concerning the nature of the infra-red divergence.

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Nuclear Physics
1 (1956)

ON THE VACUUM CURRENT

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Received 30 July 1955

Abstract: This note contains a discussion of some formulas for the vacuum current which can be derived from quantized relativistic electron theory without recourse to perturbation method.

1. Introduction

An interesting consequence of relativistic electron theory is the possibility for the vacuum to be polarized by an electromagnetic field. Polarization of the vacuum gives rise to an induced current which will be referred to as the vacuum current. It is the object of the present note to discuss some formulas for the vacuum current which can be derived without recourse to perturbation method.

We shall be concerned with the polarization of the vacuum by an unquantized electromagnetic field due to a given external source distribution. The vacuum current under consideration is the expectation value of the current operator — the operator representing the charge-current density of the electron pair field — for the polarized vacuum. There are two forms for the current operator which have the correct expectation value for the undisturbed vacuum state: the symmetrized form and the ordered form [1, 2]). Confining our attention to the time-independent fields for which separation of positive and negative energy stationary states is possible, we shall give an elementary treatment of both forms of the current operator. We shall also discuss the interaction of the vacuum current with the quantized electromagnetic field which is associated with photons.

In a later section the elementary formulas for the vacuum current will be expressed in terms of Green's functions of the Stueckelberg-Feynman type. A result for the symmetrized current operator will

† Also supported by the Nuclear Research Foundation within the University of Sydney.

be seen to be the same as a formula recently given by Schwinger [3]). We shall also obtain for both forms of the current operator, without using power-series expansions, integral representations of the vacuum current. Because of the similarity in mathematical structure, these results may be considered as exact versions of the approximate formulas which have been investigated in detail in connection with the photon self-energy and the displacement of atomic energy levels.

2. Elementary Treatment

Let E_l^0 denote the eigenvalues of stationary states of a free electron according to Dirac's one-particle theory, $\varphi_l^0(x)$ the corresponding wave functions, where x stands for the three space coordinates. Let E_l and $\varphi_l(x)$ be the eigenvalues and wave functions for an electron moving in the unquantized time-independent field A_μ of a given charge-current distribution. The eigenvalues E_l^0 are distributed continuously, but the eigenvalues E_l may be both continuous and discrete. For convenience, we shall replace continuous eigenstates by discrete ones. The wave functions φ_l and φ_l^0 are assumed to form two complete, orthonormal sets.

Let $\psi(x)$ be the quantized Dirac field. It may be expanded in terms of the wave functions φ_l or the wave functions φ_l^0, thus:

$$\psi(x) = \sum_l a_l \varphi_l(x), \tag{1}$$

$$\psi(x) = \sum_l a_l^0 \varphi_l^0(x). \tag{2}$$

The operators a obey the anticommutation rules

$$\{a_l, a_m\} = 0, \ \{a_l^\dagger, a_m^\dagger\} = 0, \ \{a_l, a_m^\dagger\} = \delta_{lm}. \tag{3}$$

The operators a^0 obey the same anticommutation rules with a replaced by a^0. The operators a^0 and $a^{0\dagger}$ for the positive-energy states are interpreted as annihilation and creation operators for a free negaton, while the same operators for the negative-energy states are interpreted as creation and annihilation operators for a free positon. Hence the vacuum state $|\Psi_v^0\rangle$ in the absence of external field satisfies the conditions

$$\begin{aligned} a_l^0 \, |\Psi_v^0\rangle &= 0 \quad (E_l^0 > 0), \\ a_l^{0\dagger} |\Psi_v^0\rangle &= 0 \quad (E_l^0 < 0). \end{aligned} \tag{4}$$

In the presence of an external field, the roles of the operators a^0

are taken over by the operators a, and the disturbed vacuum state $|\Psi_v\rangle$ satisfies the conditions

$$\begin{aligned} a_l \,|\, \Psi_v \rangle &= 0 \quad (E_l > 0), \\ a_l^\dagger \,|\, \Psi_v \rangle &= 0 \quad (E_l < 0). \end{aligned} \tag{5}$$

The two sets of operators a^0 and a correspond to the "free" and "bound" interaction representations respectively [4,5]).

For two operators a_l, a_m one can define the ordered product

$$\begin{aligned} a_l^\dagger a_m \,|_{\text{ord}} &= a_l^\dagger a_m \qquad (E_l > 0, \text{ or } E_m > 0, \text{ or both}) \\ &= -\, a_m a_l^\dagger \quad (E_l < 0, \text{ or } E_m < 0, \text{ or both}) \end{aligned} \tag{6}$$

where the two factors are ordered in such a way that the expectation value of the product vanishes for the vacuum state $|\Psi_v\rangle$. Similarly, one can define an ordered product for each pair of operators a^0.

The current operator is, in natural units,

$$\mathcal{J}_\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x) + \text{constant}, \tag{7}$$

where $\gamma_\mu = (\beta\boldsymbol{\alpha}, i\beta)$, $\bar{\psi} = \psi^\dagger \beta$, and the constant is chosen such that $\mathcal{J}_\mu(x)$ has vanishing expectation value for the undisturbed vacuum state $|\Psi_v^0\rangle$. There are two forms of the current operator which do not involve the infinite constant explicitly, namely the symmetrized form

$$\mathcal{J}_{\mu,\,\text{sym}}(x) = \tfrac{1}{2} \sum_{l,\,m} [a_l^{0\dagger}, a_m^0] \langle l^0 \,|\, j_\mu(x) \,|\, m^0 \rangle, \tag{8}$$

and the ordered form

$$\mathcal{J}_{\mu,\,\text{ord}}(x) = \sum_{l,\,m} a_l^{0\dagger} \langle l^0 \,|\, j_\mu(x) \,|\, m^0 \rangle\, a_m^0 \,|_{\text{ord}}, \tag{9}$$

where we use the notation

$$\begin{aligned} \langle l^0 \,|\, j_\mu'(x) \,|\, m^0 \rangle &= e\bar{\varphi}_l^0(x)\, \gamma_\mu \varphi_m^0(x), \\ \langle l \;\; |\, j_\mu(x) \,|\, m \;\; \rangle &= e\bar{\varphi}_l(x)\, \gamma_\mu \varphi_m(x). \end{aligned} \tag{10}$$

From eq. (8) and (9) we see that

$$\mathcal{J}_{\mu,\,\text{sym}}(x) - \mathcal{J}_{\mu,\,\text{ord}}(x) = -\tfrac{1}{2} \sum_l \sigma_l^0 \langle l^0 \,|\, j_\mu(x) \,|\, l^0 \rangle, \tag{11}$$

where $\sigma_l^0 = \text{sgn}\, E_l^0$. That the two forms are equivalent is a consequence of the equation

$$\sum_l \sigma_l^0 \langle l^0 \,|\, j_\mu(x) \,|\, l^0 \rangle = 0. \tag{12}$$

The ordering in eq. (9) ensures automatically the vanishing of the

expectation value of the current operator for the vacuum state $|\Psi_v^0\rangle$. The expectation value of the symmetrized current operator vanishes also, on account of eq. (12). In the case of the energy of free negatons and positons, however, the vacuum expectation value of the ordered form vanished, but that of the symmetrized form does not, as has been shown by Pauli [6]) and Schwinger [7]).

From eq. (8) and eqs. (1), (2), we obtain

$$\mathcal{J}_{\mu,\,\mathrm{sym}}(x) = \tfrac{1}{2}\sum_{l,\,m}[a_l^\dagger,\ a_m]\langle l\,|\,j_\mu(x)\,|\,m\rangle, \tag{13}$$

and so

$$\mathcal{J}_{\mu,\,\mathrm{sym}}(x) = \sum_{l,\,m} a_l^\dagger\langle l\,|\,j_\mu(x)\,|\,m\rangle a_m\,|_{\mathrm{ord}} + \langle\,\mathcal{J}_{\mu,\,\mathrm{sym}}(x)\,\rangle, \tag{14}$$

where

$$\langle\,\mathcal{J}_{\mu,\,\mathrm{sym}}(x)\,\rangle = -\tfrac{1}{2}\sum_{l}\sigma_l\langle l\,|\,j_\mu(x)\,|\,l\rangle \tag{15}$$

is the expectation value of $\mathcal{J}_{\mu,\,\mathrm{sym}}(x)$ for the polarized vacuum $|\Psi_v\rangle$.

The ordered form, eq. (9), provides us with the following alternative treatment of the current operator. From eq. (1), (2) we see that

$$\bar\psi(x)\gamma_\mu\psi(x) = \sum_{l,\,m} a_l^\dagger\langle l\,|\,j_\mu(x)\,|\,m\rangle a_m\,|_{\mathrm{ord}} + \sum_{l}^{(-)}\langle l\,|j_\mu(x)\,|l\rangle, \tag{16}$$

$$\bar\psi(x)\gamma_\mu\psi(x) = \sum_{l,\,m} a_l^{0\dagger}\langle l^0\,|\,j_\mu(x)\,|\,m^0\rangle a_m^0\,|_{\mathrm{ord}} + \sum_{l}^{(-)}\langle l^0\,|\,j_\mu(x)\,|\,l^0\rangle, \tag{17}$$

where we use $(+)$, $(-)$ to denote summation over positive and negative energy states only. Using these equations we obtain from eq. (9)

$$\mathcal{J}_{\mu,\,\mathrm{ord}}(x) = \sum_{l,\,m} a_l^\dagger\langle l\,|\,j_\mu(x)\,|\,m\rangle a_m\,|_{\mathrm{ord}} + \langle\,\mathcal{J}_{\mu,\,\mathrm{ord}}(x)\,\rangle, \tag{18}$$

where

$$\langle\,\mathcal{J}_{\mu,\,\mathrm{ord}}(x)\,\rangle = \sum_{l}^{(-)}\langle l\,|\,j_\mu(x)\,|\,l\rangle - \sum_{l}^{(-)}\langle l^0\,|\,j_\mu(x)\,|\,l^0\rangle \tag{19}$$

is the expectation value of $\mathcal{J}_{\mu,\,\mathrm{ord}}(x)$ for the polarized vacuum $|\Psi_v\rangle$.

Eq. (19) fits in with the original view of the positon theory [8]) that the vacuum is an infinite distribution of electrons occupying all the negative-energy states. This view provides us with a convenient interpretation of vacuum polarization [9]), in accordance with which we visualize the vacuum current as the change in the

total current of the negative-energy electrons due to the external field.

It should be mentioned here that we have treated a problem of an infinite number of dimensions as if there were a finite number of dimensions, a procedure that may lead to ambiguous results. The divergent integrals that usually appear in quantum-electrodynamical problems are a consequence of the infinite number of dimensions. There have been suggestions by Green [10]) and others that quantum electrodynamics should be reformulated in such a way that no infinity ever appears, but no attempt is made in that direction in the present note.

Up to the present we have dealt with only the unquantized electromagnetic field of a given external source. If we take into consideration the quantized electromagnetic field $A_{\mu,\,\mathfrak{p}\hbar}(x)$ which is associated with photons, there is an interaction between the electron pair field and the photon field, namely

$$\mathscr{H} = -\int \mathscr{J}_{\mu}(x)\, A_{\mu,\,\mathfrak{p}\hbar}(x)dx. \tag{20}$$

Using eq. (14), (18), we see that the interaction $\mathscr{H}$ may be separated into two terms. The first term, which is ordered, is responsible for processes such as the transition of an electron from one atomic orbit to another with emission or absorption of a photon. The second term,

$$\mathscr{H}_v = -\int <\mathscr{J}_{\mu}(x)> A_{\mu,\,\mathfrak{p}\hbar}(x)dx, \tag{21}$$

gives the interaction between the photon field and the vacuum current due to the external electromagnetic field.

3. Expressions in Terms of Green's Functions

We proceed now to express the above elementary formulas in terms of Green's functions. Let X denote the 4-vector $(x,\ t)$ and let

$$\varphi_l(X) = \varphi_l(x) \exp(-iE_l t), \tag{22}$$

$$\varphi_l^0(X) = \varphi_l^0(x) \exp(-iE_l^0 t) \tag{23}$$

be the time-dependent wave functions of the stationary states l, l^0. Let G_+ denote the Green's function

$$\begin{aligned} G_+(X,\ X') = &\quad i\sum_{l}^{(+)} \varphi_l(X)\bar{\varphi}_l(X') \quad (t > t') \\ &= -i\sum_{l}^{(-)} \varphi_l(X)\bar{\varphi}_l(X') \quad (t < t'), \end{aligned} \tag{24}$$

and G_+^0 the Green's function defined with φ^0 instead of φ. As shown by Feynman [11]) and Schwinger [3]), these Green's functions satisfy the differential equations

$$[\gamma_\mu(-i\partial_\mu - eA_\mu) + m]G_+(X, X') = \delta(X - X'), \tag{25}$$

$$(-i\gamma_\mu\partial_\mu + m)G_+^0(X, X') = \delta(X - X'), \tag{26}$$

and the integral equations

$$\begin{aligned} G_+(X,X')-G_+^0(X,X') &= e\int_{-\infty}^{\infty} G_+^0(X, X'')\gamma_\nu A_\nu(X'')G_+(X'', X')dX'' \\ &= e\int_{-\infty}^{\infty} G_+(X, X'')\gamma_\nu A_\nu(X'')G_+^0(X'', X')dX''. \end{aligned} \tag{27}$$

There are other Green's functions that satisfy the same differential and integral equations. The use of Green's functions of the above special type in quantum-electrodynamics was initiated by Stueckelberg [12]) and Feynman [11]). Green's functions of this type have also been discussed by other authors in recent years [13, 14]).

From eq. (24) we see that the function $G_+(X, X')$ shows a discontinuous jump at $t = t'$. Let us use $(+)$, $(-)$ to denote the limits as $X' \to X$ with $t - t'$ approaching zero from the positive and negative directions respectively, and $(\pm)$ the average of these limits. We have then

$$\lim_{X'\to X}{}^{(-)}\, G_+(X, X') = -i\sum_l{}^{(-)}\varphi_l(X)\bar{\varphi}_l(X), \tag{28}$$

$$\lim_{X'\to X}{}^{(\pm)} G_+(X, X') = \frac{i}{2}\sum_l \sigma_l\varphi_l(X)\bar{\varphi}_l(X), \tag{29}$$

and similar equations for G_+^0.

On account of eq. (29), (28), the expressions (15), (19) for the vacuum current may be written

$$\langle J_{\mu,\,\mathrm{sym}}(X)\rangle = ie\lim_{X'\to X}{}^{(\pm)}\mathrm{tr}\,\gamma_\mu G_+(X, X'), \tag{30}$$

$$\langle J_{\mu,\,\mathrm{ord}}(X)\rangle = ie\lim_{X'\to X}{}^{(-)}\mathrm{tr}\,\gamma_\mu[G_+(X, X') - G_+^0(X, X')] \tag{31}$$

respectively, where we write X instead of x for the sake of uniformity in notation. Eq. (30) has been given by Schwinger. On account of the equation

$$\lim_{X'\to X}{}^{(\pm)}\,\mathrm{tr}\,\gamma_\mu G_+^0(X, X') = 0, \tag{32}$$

which is equivalent to eq. (12), eq. (30) may be written in the

same form as eq. (31). These equations give us therefore, with the help of the integral eq. (27),

$$\begin{aligned}\langle \mathscr{J}_\mu(X) \rangle &= ie^2 \lim_{X' \to X} \int_{-\infty}^{\infty} \operatorname{tr} \gamma_\mu G_+^0(X, X'') \gamma_\nu A_\nu(X'') G_+(X'', X') dX'' \\ &= ie^2 \lim_{X' \to X} \int_{-\infty}^{\infty} \operatorname{tr} \gamma_\mu G_+(X, X'') \gamma_\nu A_\nu(X'') G_+^0(X'', X') dX''.\end{aligned} \tag{33}$$

These formulas are similar in mathematical structure to the usual approximate formula

$$\langle \mathscr{J}_\mu(X) \rangle = ie^2 \int_{-\infty}^{\infty} \operatorname{tr} \gamma_\mu G_+^0(X, X') \gamma_\nu A_\nu(X') G_+^0(X', X) dX'. \tag{34}$$

Our analysis shows that one can prescribe different limiting forms for the vacuum current. However, because of the singularity of the Green's functions on the light-cone, the difference between the limiting processes is rather formal. It is well known that the approximate expression (34) for the vacuum current may be separated into a divergent, non-physical part and a unique, finite part which is physically significant. It is to be expected that a similar situation exists for the exact value of the vacuum current.

The writer is grateful to Professor H. Messel for his encouragement and interest.

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Nuclear Physics
1 (1956)

ON THE ANOMALOUS SCATTERING OF μ-MESONS

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Received 12 July 1955

Abstract: The inelastic scattering of μ-mesons with excitation of the "giant" nuclear dipole resonance and its influence on the large angle scattering of μ-mesons are discussed. It is concluded that the Williams virtual photon method in its original form is not applicable to problems of this kind and that the excitation of "giant" dipole resonances is unable to explain the observed large angle scattering.

1. Introduction

In recent years several authors, for example Whittemore and Shutt [1]), Kannangara and Shrikantia [2]), and Leontic and Wolfendale [3]), have reported a cross section for large angle scattering of high energy cosmic ray μ-mesons which is an order of magnitude larger than that to be expected from the Coulomb force between the incident particle and an extended nucleus. The energies of the μ-mesons in question are larger than 200 MeV and the angles of deflection, θ, are such that $p\,\theta > 5(\mathrm{GeV}/c)$ degrees, where p is the momentum of the incident meson.

In order to explain this effect different authors [4,5]) have calculated the elastic scattering to the first order of Born's approximation and the inelastic scattering to the same approximation using a single particle nuclear model. It is found that the results are too small to fit the experimental observations at large angles.

It has recently been suggested, however, by Davidson [6]) that the "giant" nuclear dipole resonances, observed when nuclei are irradiated by gamma rays of 10—20 MeV energy [7]), may be responsible for the large angle scattering. However, Davidson, using Williams' method [8]), only considered the total cross sections. In the following note we examine his suggestion in more detail by calculating the differential cross sections, and conclude that

the "giant" resonances are not able to account for the observed large angle scattering.

2. Calculation of the Differential Cross Section

According to the method due to Møller [9]) a scattered μ-meson may be represented, to the first Born approximation, by an equivalent electromagnetic field under the influence of which the nucleus undergoes a transition.

At this point it is necessary to assume an interaction mechanism and in the present note, following Foldy [10]), we shall assume that the interaction, whatever its nature, is expressible in terms of the nucleon coordinates alone in the non-relativistic approximation for the nucleons. Taking the "one particle" part (in Foldy's terminology) of the nuclear charge and current distribution we will write for the interaction term in the Hamiltonian of the system

$$-\sum_n \mathbf{j}_n \cdot \mathbf{A}_\mu + \sum_n \frac{e_n e_\mu}{r_{n\mu}}$$

where the suffix n refers to the nucleons, the suffix μ to the meson and $r_{n\mu}$ is the nucleon-meson distance.

If the wave function of the incident particle is written as

$$\psi = \varphi(p) e^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$$

and we put

$$a_0 = \varphi^*(p')\varphi(p),$$
$$\mathbf{a} = \varphi^*(p')\boldsymbol{\alpha}\varphi(p),$$

then the cross section for scattering into the element of solid angle $d\omega$ with the transfer to the nucleus of an energy W and a momentum $\mathbf{q}$ is given by

$$\sigma_{if} = \frac{4\alpha^2 p' E E' d\omega}{p(q^2 - W^2)^2} \left| \sum_n \int \psi_f^* e^{i\mathbf{q}\cdot\mathbf{r}_n} (a_0 + \mathbf{a}\cdot\boldsymbol{\alpha}_n)\psi_i d\tau \right|^2. \qquad (1)$$

Here ψ_i and ψ_f are the initial and final nuclear state wave functions, p' and E' refer to the μ-meson after the collision, α is the fine structure constant and the units used are such that $\hbar = M_\mu = c = 1$.

3. Discussion of Differential Cross Section

To make the argument clear it is convenient to discuss separately two approximations appropriate to the cases of scattering through

angles such that $p\theta \ll 1/R$ and $p\theta \gg 1/R$, respectively, where R denotes the nuclear radius. In each instance we shall find it possible to introduce experimental results on nuclear photo-absorption and find an upper limit to the cross section.

(i) $p\theta \ll 1/R$

For angles such that $p\theta \ll 1/R$, the result (1) may be written

$$\sigma_{if} = \frac{4\alpha^2 p' E E' d\omega}{p(q^2 - W^2)^2} \left| \sum_n \int \psi_f^* \mathbf{r}_n \psi_i d\tau \right|^2 |\mathbf{q} - W\mathbf{p}/E|^2. \tag{2}$$

For the heavier nuclei, W will be greater than q and the cross section, being given by

$$\sigma_{if} \approx \frac{4\alpha^2 p' E E' dw}{pW^2} \cdot \left| \sum_n \int \psi_f^* \mathbf{r}_n \psi_i d\tau \right|^2, \tag{3}$$

will be independent of angle.

We may introduce the experimental results into (3) by using the following relation:

$$\left| \sum_n \int \psi_f^* \mathbf{r}_n \psi_i d\tau \right|^2 = \frac{f_{if}}{2M_p(E_f - E_i)} = \frac{\sigma_{h\nu} dW}{4\pi^2 e^2 W}. \tag{4}$$

Here $\sigma_{h\nu}$ is the photo-absorption cross section, f_{if} the oscillator strength for the transition $i \to f$ and M_p the proton mass. By summing over the oscillator strengths f_{if} we may set an upper limit to the cross section. Thus (writing σ for any σ_{if})

$$\sigma < \frac{\alpha p' E E' dw}{\pi^2 p W^3} \int_0^\infty \sigma_{h\nu} dW.$$

According to Levinger and Bethe [11]) the best value to assign to the quantity $\int_0^\infty \sigma_{h\nu} dW$ is (in the present units)

$$\int_0^\infty \sigma_{h\nu} dW = 0.04 \frac{NZ}{A}. \tag{4'}$$

Thus we see that, for $p\theta \ll 1/R$, the differential cross section is independent of θ and satisfies the inequality

$$\sigma \leqq \frac{0.04\, \alpha^2 p' E E' \cdot NZ dw}{\pi^2 p W^3 A}. \tag{5}$$

(ii) $p\theta \gg 1/R$

Here the dipole approximation is not valid and we have

$$\sigma_{if} \approx \frac{4\alpha^2 p' EE'}{pq^4} \left| \sum_n \int \psi_f^* e^{iq.r_n} \psi_i d\tau \right|^2. \tag{6}$$

We have assumed that $q > W$ and that $W^2 \ll 1$. These conditions are satisfied in the present application.

With a more general definition of oscillator strength due to Bethe [12]), namely,

$$\left| \sum_n \int \psi_f^* e^{iq.r_n} \psi_i d\tau \right|^2 = \frac{q^2 f_{if}(q)}{2M_p(E_f - E_i)}, \tag{7}$$

we may set an upper limit to the cross section in the same manner as before. If there were no exchange couplings in the nuclear interaction we could use the sum rule [12])

$$\sum_n f_{if}(q) = Z,$$

which would give us an upper limit independent of q. However, we must make allowance for the effect of exchange forces, and this may depend on q in a way which would be cumbersome to evaluate. We shall simply take over the correction factor 1.6 indicated by the experimental results for energies $\approx$ 20 MeV. This gives

$$\sigma \approx \frac{6.4\,\alpha^2 p' EE' Z dw}{2pq^2 W M_p}. \tag{8}$$

4. Comparison with the Observed Anomalous Cross Section

The experimental results on the anomalous large angle scattering of μ mesons have been found to coincide approximately with that predicted when the nucleus is considered to be a point charge of magnitude Ze. That is to say

$$\sigma_{\text{exp}} \approx \frac{4 Z^2 \alpha^2 E^2 dw}{q^4}.$$

The ratio of the result (8) to the experimental observations on anomalous scattering is thus

$$\frac{\sigma_{\text{g.d.}}}{\sigma_{\text{exp.}}} \approx \frac{1.6\, q^2}{2WZM_p}. \tag{9}$$

For values of $q \approx 10$ (GeV/c) · deg this gives

$$\frac{\sigma_{\text{g.d.}}}{\sigma_{\text{exp}}} \approx \frac{1.6}{Z}, \tag{10}$$

which shows that the existence of the giant resonances in no way accounts for the experimental results.

As a matter of fact the result (9) is even a gross overestimate. This may be seen from the following argument. For $qR \gg 1$ the matrix element in (6) is only appreciable when the final state is a sum of terms of the form

$$\psi_f \approx e^{i\mathbf{P}\cdot\mathbf{r}}\varphi\ (Z-1),$$

where the function φ depends on the coordinates of $Z-1$ protons and $\mathbf{P} = \mathbf{q}$. For such cases W is given approximately by

$$W = E_f - E_i \approx \frac{q^2}{2M_p}. \tag{11}$$

Substituting from (7) and (11) in (6) we find the following upper limit to the cross section:

$$\sigma \leqq \frac{4\alpha^2 p' E E' Z}{pq^4}. \tag{12}$$

This result coincides with that given by Gatto [5]) on the basis of a detailed calculation using a single particle nuclear model. It corresponds to the case of Z nuclear particles scattering independently, the nuclear forces playing only a minor role. Conversely, resonance transitions between the ground and excited states of the nucleus as a whole must be rare in these circumstances. Thus for $qR \gg 1$ the 'giant' dipole matrix element must be very small and the differential cross section several orders of magnitude too small to account for the anomalous scattering.

5. The Williams Method

The Williams method in the form used by Davidson is not applicable to problems of this kind. The reason for this lies in the assumption, basic to the method, that the perturbing electric field due to the incident particle is uniform over the perturbed system. From this assumption it follows that we can write the transition probability as a product of two terms, one being a virtual photon flux, the other being the photo-absorption cross section. The cross section then reduces to the dipole approximation of eq. (2). For scattering through angles such that $qR > 1$ this is obviously inadequate. This

question has already been discussed in more detail in a previous paper [13]).

The author is indebted to Drs. G. D. Rochester and A. W. Wolfendale and to Dr. I. McDiarmid for many interesting discussions.

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Nuclear Physics
1 (1956)

ON THE ANOMALOUS MAGNETIC MOMENT OF THE μ-MESON

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Recived 12 July 1955

Abstract: A discussion is given of various phenomena involving μ-mesons, especially bremsstrahlung and pair production, in which an anomalous magnetic moment of the meson should be important, taking into account finite meson size effects in a relativistic way. It is found that the empirical evidence rules out the existence of an anomalous magnetic moment of any significance for the μ-meson.

1. Introduction

Recent experiments on the scattering of μ-mesons by nuclei [1]) have disclosed a larger cross section for scattering through large angles than that predicted on the basis of the normal electromagnetic interaction taking into account the finite size of the nucleus. The momentum transfers involved are of the order $p\theta \approx 5$ (GeV/c) · deg, where p is the momentum of the incident meson and θ the angle of deflection.

Various explanations of this result have been proposed; in particular, it has been suggested that the μ-meson may have a sufficiently large anomalous magnetic moment to give the required cross section at large scattering angles [2, 3]).

We can estimate such an effect by noting the r^{-2} dependence of the magnetic interaction potential. If σ_λ is the cross section for scattering through an angle θ due to the anomalous magnetic moment of λ Bohr magnetons alone, and σ_n the usual scattering cross section, then we find for the ratio of the magnetic to the ordinary scattering

$$\frac{\sigma_\lambda}{\sigma_n} \approx \frac{\lambda^2 (p\theta)^2}{30} \qquad (p\theta \text{ in (GeV/}c\text{)} \cdot \text{degrees}).$$

When $p\,\theta > 6$ the experimental results are approximately an order of magnitude larger than the predictions of the usual theory based on the normal electromagnetic interaction. Thus we require $\lambda^2 \geqq 10$. This is confirmed by the detailed calculations of Gatto [3]).

However, as is mentioned by Gatto, it is necessary to note, that, associated with an anomalous magnetic moment, we must expect a meson cloud which will give rise to a cut off in the momentum transfer when the finite size of the μ-meson is taken into account. Exact evaluation of this effect depends on the type of meson form factor assumed but it should be expected, that the experimentally observed momentum transfers would be excluded by any reasonable choice.

In view of this situation it is of interest to investigate the influence of an anomalous magnetic moment on other electromagnetic phenomena, taking into account finite size effects, with a view to ascertaining the restrictions on the magnitude of λ imposed by experimental results.

A previous discussion of this question has been given by Peaslee [4]) but in that work the finite size effects were not taken into account in a consistent way. In the present note, we use the relativistic procedure given by Corinaldesi [5]).

The electromagnetic phenomena of interest besides scattering are the following:

(i) Ionisation loss
(ii) Bremsstrahlung
(iii) Pair production
(iv) Underground star production
(v) Fine structure of the μ-mesic atom.

So far as the last named is concerned the present experimental results are not sufficiently accurate to say more [6]) than that $\lambda < 8$ and we will therefore not discuss it further.

2. Relativistic cut-off Procedure

The relativistic procedure just referred to consists in writing for the anomalous magnetic part of the μ-meson current density

$$J^m_\mu(x) = \frac{1}{(2\pi\hbar c)^4} \int e^{ip\cdot(x-X)/\hbar c}\, \mathcal{J}^m(p^2)\, \frac{\partial m_{\mu\nu}(X)}{\partial X_\nu}\, d^4X d^4p, \qquad (1)$$

where

$$m_{\mu\nu} = \tfrac{1}{2}\lambda\bar{\psi}\sigma_{\mu\nu}\psi, \qquad (2)$$

with

$$\sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2i,$$

is the density tensor of anomalous magnetisation and $\mathcal{J}^m(p^2)$ the Fourier transform of the form factor describing the assumed finite

size of the meson magnetons; a point particle of anomalous moment λ is described by

$$\mathcal{J}^m(p^2) \equiv 1.$$

It can be shown that the matrix element occurring in the description of a collision process are multiplied by a factor $\mathcal{J}^m((\Delta p)^2)$ where Δp is the four momentum transfer.

We may suppose that

$$\mathcal{J}^m(p^2) = e^{-\left|\frac{p^2\lambda_\mu}{\hbar^2}\right|}, \tag{3}$$

where $\lambda_\mu = \hbar/m_\mu c$. The most convenient system in which to express Δp is the centre of mass system; we have in this system

$$\Delta p = (\mathbf{p} - \mathbf{p}', 0).$$

From (3) the largest momentum transfer for which the cross section is appreciable is then

$$\Delta p_{\max} = \hbar/\lambda_\mu \text{ or } p\theta \approx 6 \text{ (GeV}/c) \cdot \text{deg}. \tag{4}$$

3. Discussion

3.1. IONISATION LOSS

When the energy of the incident particle is large compared with its rest energy, the cross section for energy transfer E to an atomic electron neglecting atomic binding is given, in the laboratory system, by

$$d\sigma = \frac{\pi e^4 \lambda^2}{(m_\mu c^2)^2} \cdot \frac{df}{f}(1 - f), \tag{5}$$

with $f = E/E_0$, where E_0 is the energy of the incident particle [7,8]).

The four momentum cut off, which reduces to (4) in the centre of mass system, gives an upper limit to f of $f_{\max} = m_\mu^2 c^2/2E_0 m_e$. We then find for the ionisation energy loss per unit length of path when N is the number of electrons per cm^3

$$\Delta = E_0 \int_0^{f_{\max}} f d\sigma \approx \frac{\pi r_\mu^2 N m_\mu^2 c^2}{2m_e}, \tag{6}$$

where $r_\mu = e^2/m_\mu c^2$. In deriving this result it has been assumed that $E_0 > (m_\mu/m_e)\, m_\mu c^2$.

This result is the same as that given by Peaslee, who in fact also uses in this case a cut-off equivalent to (4). However, in the other processes he discusses (bremsstrahlung and pair production) he assumes a formula of type (4) in the rest system of the meson. Had he used this prescription in the present case he would not have found an upper limit to the energy loss since the two systems considered are the same only in the limit $m_e/m_\mu \to 0$.

The result (6) gives a maximum energy loss per unit path length due to an anomalous magnetic moment of the order 10^{-2} MeV · cm²/g when $\lambda^2 \approx 10$. The normal ionisation loss is of the order 2 MeV · cm²/g, so that, on this evidence, this value of λ^2 cannot be excluded.

3.2. BREMSSTRAHLUNG

The production of bremsstrahlung using the relativistic procedure adopted to take into account the finite size of the meson way be treated in the following way. With

$$\psi(X) = \frac{1}{(2\pi\hbar c)^{3/2}} \int d^4p' \psi(p') e^{ip' \cdot X/\hbar c}$$

and

$$\bar{\psi}(X) = \frac{1}{(2\pi\hbar c)^{3/2}} \int d^4p'' \bar{\psi}(p'') e^{ip'' \cdot X/\hbar c},$$

The expression for the anomalous contribution to the current density becomes, after partial integration with respect to X,

$$J^m_\mu(x) = \frac{1}{(2\pi\hbar c)^3} \frac{\lambda}{2} \int d^4p d^4p' d^4p'' \bar{\psi}(p'') \sigma_{\mu\nu} \psi(p') \cdot \mathcal{J}^m(p^2) \cdot e^{i(p'-p'')X/\hbar c} e^{ip(x-X)/\hbar c} (-ip_\nu) d^4X. \quad (7)$$

Integrating over X and p gives

$$J^m_\mu(x) = \frac{1}{(2\pi\hbar c)^3} \frac{\lambda}{2} \int d^4p' d^4p'' \bar{\psi}(p'') \sigma_{\mu\nu} \psi(p') e^{i(p'-p'')x/\hbar c} \cdot \cdot (-i(p'-p''))_\nu \mathcal{J}^m((p'-p'')^2). \quad (8)$$

The matrix element for bremsstrahlung is of the form

$$K_2 = -\frac{i}{2\pi\hbar^2} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 \frac{1 + \varepsilon(t_2 - t_1)}{2} H(t_2) H(t_1), \quad (9)$$

where

$$H(t) = -\int J^m_\mu(x) A_\mu(x) d\tau$$

and

$$\varepsilon(t_2 - t_1) = +1, t_2 > t_1$$
$$= -1, t_2 < t_1.$$

Using (8) and the Fourier transform of $A_\mu(x)$, this gives

$$K_2 = \frac{-i}{2\hbar^2} \cdot \frac{1}{(2\pi\hbar c)^9} \int d^4x_2 d^4x_1 d^4p_2' d^4p_2'' d^4p_1' d^4p_1'' d^4k_2 d^4k_1 \cdot$$
$$\cdot \frac{1 + \varepsilon(x_{02} - x_{01})}{2} e^{i(s_2 \cdot x_2 + s_1 \cdot x_1)/\hbar c}$$
$$\cdot \bar{\psi}(p_2'')\sigma_{\mu\nu}\psi(p_2')(-i(p_2' - p_2''))_\nu A_\mu(k_2)\bar{\psi}(p_1'')\sigma_{\varrho\varkappa}\psi(p_1')$$
$$\cdot (-i(p_1' - p_1''))_\varkappa A_\varrho(k_1)\mathcal{J}^m((p_2' - p_2'')^2)\mathcal{J}^m((p_1' - p_1'')^2) \quad (10)$$

where

$$s_i = p_i' + k_i - p_i''.$$

Pairing off the wave functions representing emission and absorption of virtual particles and taking the expectation value for the meson vacuum introduces, instead of $\psi(p_2')\ \bar{\psi}(p_1'')$, the expression (Heitler [9]), p. 285)

$$S\langle\psi_\varrho(p_2')\bar{\psi}_{\varrho'}(p_1'')\rangle_0 = T_{\varrho\varrho'}\tfrac{1}{2}(1+\varepsilon(p_{20}')\varepsilon(x_{02}-x_{01}))\delta^4(p_2' - p_1'')\delta(p_2'^2+\mu^2),$$

where S refers to a summation over spin states of the virtual particles and

$$T_{\varrho\varrho'} = \gamma_{\varrho\varrho'} \cdot p_2' + i\mu.$$

The factor $\varepsilon(x_{02} - x_{01})$ arises when we take account of the permutation of the emission and absorption operators of the virtual particles. The factor

$$(1 + \varepsilon(p_{20}')\varepsilon(x_{02} - x_{01}))\delta(p_2'^2 + \mu^2)$$

has the same effect under the integral sign (in the present case) as $\delta_+(p_2'^2 + \mu^2)$ (Heitler [9]), p. 286) and when we sum over all permutations of the points x_1, x_2, using

$$\tfrac{1}{2}\sum_P P(1 + \varepsilon(x_{02} - x_{01})) = 1,$$

the integrand of (10) depends on the x_i only through the factor $\exp(is_i \cdot x_i/\hbar c)$. The integrations over the x_i may now be carried out giving the usual Dirac functions $(2\pi\hbar c)^4\delta^4(s_i)$.

If now we take plane waves for the incident and outgoing mesons:

$$\psi_\varrho(p_1') = \delta^4(P - p_1')\, a_\varrho(P),$$
$$\bar{\psi}_\varrho(p_2'') = \delta^4(P' - p_2'')\, a_\varrho(P'),$$

we may carry out the integrations over p_1', p_1'' and p_2''. The result may be written, before integrating over k_1,

$$dK_2 = \frac{i}{2\hbar^2} \cdot \frac{1}{2\pi\hbar c} \int d^4p_2' d^4k_2 \delta^4(p_2' + k_2 - P')\delta^4(P + k_1 - p_2') \cdot a(P')\,\sigma_{\mu\nu} T A_\mu(k_2)\sigma_{\varrho\varkappa} a(P) \cdot (p_2' - P')_\nu (P - p_2')_\varkappa \qquad (11)$$
$$A_\varrho(k_1)\, \mathcal{J}^m(k_2^2)\, \mathcal{J}^m(k_1^2).$$

The δ functions in (11) imply

$$k_2 = P' - P - k_1$$

and

$$p_2' = P + k_1.$$

The remaining integrations may be carried out to give

$$dK_2 = -\frac{i}{2\hbar^2} \cdot \frac{1}{2\pi\hbar c} \cdot a(P')\sigma_{\mu\nu} T A_\mu(P' - P - k_1)\sigma_{\varrho\varkappa} a(P) \cdot (k_1 + P - P')_\nu (k_1)_\varkappa \cdot A_\varrho(k_1)\, \mathcal{J}^m\big((P' - P - k_1)^2\big)\, \mathcal{J}^m(k_1^2). \qquad (12)$$

Possible restrictions on four momentum transfers are contained in the terms $\mathcal{J}^m\big((P'-P-k_1)^2\big)$, $\mathcal{J}^m(k_1^2)$ and also in $A_\mu(P'-P-k_1)$. This last term sets an upper limit to momentum transfer since it includes a cut off introduced by the finite nuclear size. This leads to

$$|(\mathbf{P}' - \mathbf{P} - \mathbf{k}_1)| < \hbar/R, \qquad (13)$$

where R is the nuclear radius. The term $\mathcal{J}^m\big((P' - P - k_1)^2\big)$ introduces an effect due to the finite size of the meson when the meson and nucleus exchange a quantum. This obviously yields a less restrictive condition than (13) due to the fact that $R > \lambda_\mu$. For the remaining term we have

$$\mathcal{J}^m(k_1^2) = \mathcal{J}^m(0) = 1.$$

There is thus no cut off in radiation loss other than that introduced by the finite nuclear size.

We can therefore simply use the formula derived by Powell [20] for this case. The ratio of total radiation loss due to anomalous effects to the normal radiation loss is, according to Powell,

$$\frac{\varphi_\lambda}{\varphi_n} \approx 0.1 \frac{\lambda^4 E_0}{m_\mu c^2}. \qquad (14)$$

On the other hand Peaslee's cut off prescription led him to the essentially different expression

$$\varphi_\lambda/\varphi_n \approx 0.1\, \lambda^4.$$

The observations on cosmic ray bursts (soft showers initiated by high energy bremsstrahlung produced by mesons with energies of the order 10^3 GeV) agree approximately with the results predicted by the usual theory [11]). Thus (14) implies $\lambda^2 \ll 1$ in contrast to Peaslee's result and this conclusion is independent of the form factor specifying the finite size of the meson.

3.4. PAIR PRODUCTION

Without going into the details of the calculation it is easy to see that, as the photon responsible for the pair production is virtual, it follows that $k_1^2 \neq 0$ and

$$\mathcal{J}^m(k_1^2) < 1.$$

However, when the energy of the incident meson is sufficiently high the four momentum of the virtual photon will satisfy $k_1^2 = 0$ to order $m_\mu c/p$, where p is the momentum of the incident meson. There will then be no cut off in electron pair energy. By comparison with the bremsstrahlung case we may infer that the ratio of the cross sections for pair production will be again of order $\lambda^4 E_0/m_\mu c^2$. Now, the experimental work of Goldsack and Kannangara [12]) has shown that the observed pair production is in reasonable agreement with the usual theory, so that again $\lambda^2 \ll 1$ independently of the form factor.

3.5. STAR PRODUCTION

In this case we have to do with collisions between the incident meson and a nuclear proton, and so we have as an upper limit to the possible energy transfers

$$W_{\max} = \tfrac{1}{2} m_\mu c^2 \left(\frac{m_\mu}{M_p}\right) \approx 10 \text{ MeV}.$$

Since nuclear stars have energies larger than 300 MeV we may say that such processes are unlikely to give any information on the value of λ.

4. Conclusions

The conclusion from this discussion is that $\lambda^2 \ll 1$ independently of the finite meson size effects, so that there should be no significant contribution to the scattering of μ-mesons by nuclei from an anomalous magnetic moment.

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BOOK REVIEWS

L. I. SCHIFF, *Quantum Mechanics*. Second edition. McGraw Hill Book Co, New York, Toronto, London. 1955. xii-417 pages. $ 6.50.

PH. PLUVINAGE, *Eléments de Mécanique Quantique*. Masson et Cie, Paris. 1955. xii-547 pages. 4.600 French fr.

These two textbooks on quantum mechanics deserve some mention in view of their general excellence. Schiff's wellknown treatise now appears in a second edition which is substantially the same as the first. In particular the author has wisely resisted the temptation to replacing the account of quantum electrodynamics on traditional lines by a presentation of the new-fangled covariant techniques, which do not belong in an introductory book of this type.

Professor Pluvinage has provided French-speaking students with a much-needed textbook of outstanding quality. In fact, his book may claim universal interest in view of the elegance and lucidity of the exposition.

In neither of these books, however, do the foundations of quantum mechanics receive adequate treatment. Schiff's discussion is much too condensed to be really enlightening. With regard to Pluvinage, the cartesian spirit of which the French are so proud (according to the publisher's prospectus the French reader is "cartésien de nature"!) has played him tricks. The analogy with classical statistical mechanics which he takes as his starting-point is of course quite misleading and in any case unsuited to illustrate the dialectical, utterly non-cartesian character of quantum theory.

L. R.

A. LANDÉ, *Foundations of Quantum Theory*, a study in continuity and symmetry. Yale University Press, New Haven. 1955. viii-106 pages. $ 4.00.

Would this small book by a distinguished pioneer of quantum theory provide the needed introduction into the foundations of this theory? Most unfortunately this question must be definitely answered in the negative. The intrusion of metaphysics into epistemological problems, which ought to be treated in a rigorously scientific way, seems to be a plague of our times. There are those cartesian physicists who take Determinism as their creed. Professor Landé chooses to invoke, of all people, Leibniz, and invites us to be 'guided by the creed that the world not only is the best . . . but also the simplest of all possible worlds'. He would reduce the infinite richness of atomic laws to the shadowy Leibnizian principles of continuity and symmetry. When he ventures so far into thin air even the best of all possible physicists must lose control of reality. Professor Landé has, I am afraid, made a muddle of a perfectly clear situation.

The first confusion arises when he suggests a connection between the statistical aspects of quantum mechanics and thermodynamics. There is of course no such connection in reality; Professor Landé's considerations are of a purely formal nature and they do not receive any concrete contents simply by the use of the word 'entropy'. Moreover, the alleged principle of continuity of entropy by which the author wants to remove Gibbs' paradox in thermodynamics would be tantamount to denying the very existence of atoms and molecules; and the author's ingenious attempt to establish a formal connection between this principle and the idea of probability amplitude in quantum mechanics involves too much arbitrariness to be convincing.

As regards the symmetry principle and the restriction of identical particles to symmetrical or anti-symmetrical states, the situation is hardly better. In any

case the removal of the N!-fold degeneracy has nothing to do with the removal of Gibbs' paradox, as the author contends.

I should not like to conclude this review in such a negative mood. The real introduction to the foundations of quantum theory does exist. It is Niels Bohr's article 'Discussion with Einstein on epistemological problems in atomic physics', in the collective volume *Albert Einstein: philosopher-scientist* (The Library of Living Philosophers, Evanston 1949).

L. R.

R. G. SACHS, *Nuclear Theory*. Addison-Wesley Publ. Co., Cambridge, Mass. 1953. xi-383 pages. $ 7.50.

An excellent treatise which has lost nothing of its usefulness since its publication two years ago. It is most suitable as a guide for theoretical physicists intending to specialise in nuclear physics.

L. R.

E. FEENBERG and G. E. PAKE, *Notes on the quantum theory of angular momentum*. Addison-Wesley Publ. Co., Cambridge, Mass. 1953. vii-56 pages. $ 2.00.

This is a most useful compendium of the concepts and formulae concerning angular momentum which are of constant use in many parts of atomic and nuclear theory. The size is such that addicts of the rotation group can conveniently carry it in their pocket for reading in buses and trains.

L. R.

C. H. TOWNES and A. L. SCHAWLOW, *Microwave Spectroscopy* (McGraw-Hill Book Company, New York, Toronto, London. 1955. xviii-698 pages. 90 s. 6 d.)

In the last ten years radio-frequency spectroscopy has expanded enormously, and has yielded quantities of new information about the structure of nuclei, atoms, molecules and the solid state. The principal divisions of radiofrequency spectroscopy are: nuclear magnetic resonance, electronic magnetic resonance, and microwave gas spectroscopy. Under suitable conditions each of these is capable of very high resolution, giving great precision in the determination of interaction constants in which nuclear moments — magnetic dipole and octupole, electric quadrupole — are involved. (Except in the simplest atoms, a comparable precision cannot be achieved in the determination of many of the nuclear moments themselves because the electronic quantities involved cannot be evaluated with sufficient accuracy). A second advantage of radio-frequency methods is their high sensitivity, which makes it possible to work with radio-active isotopes at levels of activity which can be safely handled by ordinary laboratory techniques.

The methods used in the different branches of radio-frequency spectroscopy are so divergent that no book of normal size can cover them all except at an elementary level. In "Microwave Spectroscopy" the authors have wisely confined their attention to one branch, work on gaseous molecules, and the thoroughness of the treatment is best indicated by an outline of the book.

In the majority of cases, the lines observed in microwave gas spectroscopy are due to electric dipole transitions between rotational levels of molecules (these must be polar molecules for such transitions to be observable). The first four chapters therefore give the theory of these levels and the allowed transitions for linear, symmetric and asymmetric top molecules. For the nuclear physicist the principal item of interest here is the determination of isotopic mass ratios (with a precision comparable with that of the best mass spectrometers) from the fre-

quencies of rotational lines, which depend on the moment of inertia of the molecule. A chapter on atomic spectra then serves as an introduction to hyperfine structure in molecular spectra. Interaction with nuclear electric quadrupole moments is considered first, since magnetic hyperfine structure arises only in the rare molecules with a resultant electronic angular momentum. The section is rounded off by a chapter on the interpretation of hyperfine constants in molecules. The Stark and Zeeman effects, the ammonia spectrum and hindered rotation, and line width form the next section; the experimental portion consists of five chapters on spectrographs and allied techniques. Besides use in chemical analysis, the realization of a microwave frequency standard based on an atomic or molecular spectral line is outlined. A device with the ugly name of "maser" (microwave amplification by stimulated emission of radiation) using ammonia promises strong competition with the caesium atomic beam method for choice as the practicable frequency standard. The last 180 pages consist of tables in the form of seven appendices and a bibliography of over 1,000 references.

The standard of printing and of the diagrams is high, as we have come to expect of the McGraw-Hill company, and this reviewer has detected few misprints and errors (*A* for Ä on p. 299, and wrong values of the nuclear spins of the odd samarium isotopes in Appendix VII). There is no doubt that this will be the standard book on the subject for some time to come.

B. Bleaney

NEWS ITEM

INTERNATIONAL CONFERENCE ON NUCLEAR REACTIONS

AMSTERDAM, 2-7 JULY 1956

The Netherlands Physical Society (N.N.V.) organises, under patronage of the I.U.P.A.P., an INTERNATIONAL CONFERENCE on NUCLEAR REACTIONS, to be held in Amsterdam, 2—7 July 1956.

Provisional Programme of the Meetings

General introduction	H. A. Bethe
Low energy processes, experimental	
a) initiated by charged particles	
low energy proton scattering	R. F. Christy
medium energy proton scattering	P. C. Gugelot
Coulomb excitation	T. Huus
b) initiated by neutrons	
scattering of neutrons	D. Hughes
Theoretical models for low energy processes	
formation of the compound nucleus	V. Weisskopf
excitation of collective states	A. Bohr
Capture and photonuclear reactions	
capture reactions	P. M. Endt
photonuclear reactions	D. H. Wilkinson
Stripping and pick-up reactions	
experimental work	J. R. Holt
theory of direct interaction	J. Horowitz
High energy scattering (including polarisation effects)	
scattering of nucleons by nucleons and complex nuclei	E. Segrè
quasi-optical model	L. van Hove
Fission, spallation, reactions initiated by heavy ions	
spallation and heavy ion reactions	G. T. Seaborg
fission	J. A. Wheeler

It is intended to exclude the discussion of beta-processes, meson-processes and accelerating machines.

The main speakers, mentioned in this programme, have been (or will be) invited to give an introductory lecture. Each of these introductory lectures will be followed by a discussion for which short communications and remarks are invited. Participants intending to present short communications and discussion remarks at this conference should write to the Conference Committee's secretary, Professor S. A. Wouthuysen, Zeeman laboratorium, Plantage Muidergracht 4, Amsterdam, Netherlands.

FOR PROSPECTIVE CONTRIBUTORS

Since the nomenclature in the field of Nuclear Physics is not yet quite unified and a number of illogical expressions are unfortunately still widely used, it is advisable to give the following indications about the terminology adopted in this journal.

(1) The word 'electron' applies to particles with an elementary charge of either sign. When it is desirable to emphasize the sign of the charge, the words 'positon' and 'negaton' are used. Thus the illogical phrase 'electron-positron pair' is replaced by 'electron pair', or, if required, by 'positon-negaton pair'.

(2) Nuclei (or nucleids, *not* 'nuclids') are conveniently divided according to the parity of the proton and neutron numbers Z, N, into four families as follows:

even mass nuclei	even nuclei (Z even, N even) odd nuclei (Z odd, N odd)
odd mass nuclei	odd proton nuclei (Z odd, N even) odd neutron nuclei (Z even, N odd).

(3) The expressions 'isotopic spin', 'isotopic quantum number' are replaced by 'isobaric spin', 'isobaric quantum number'.

(4) For the elementary particles use is made of the latest nomenclature adopted by international agreement.

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